Published for SISSA by Springer

Received: *February 22, 2019*
Accepted: *April 28, 2019*
Published: *May 22, 2019*

Scattering equations and a new factorization for amplitudes. Part II. Effective field theories

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Abstract: We continue the program of extending the scattering equation framework by Cachazo, He and Yuan to a double-cover prescription. We discuss how to apply the double-cover formalism to effective field theories, with a special focus on the non-linear sigma model. A defining characteristic of the double-cover formulation is the emergence of new factorization relations. We present several factorization relations, along with a novel recursion relation. Using the recursion relation and a new prescription for the integrand, any non-linear sigma model amplitude can be expressed in terms of off-shell three-point amplitudes. The resulting expression is purely algebraic, and we do not have to solve any scattering equation. We also discuss soft limits, boundary terms in BCFW recursion, and application of the double-cover prescription to other effective field theories, like the special Galileon theory.

Keywords: Differential and Algebraic Geometry, Scattering Amplitudes, Sigma Models

ArXiv ePrint: 1902.02633

https://doi.org/10.1007/JHEP05(2019)129

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1 Introduction

The S-matrix elements of gravity, gauge theories and various scalar theories can be calculated using the novel scattering equation framework by Cachazo, He and Yuan (CHY) [1–3]. The n-point scattering amplitude in the CHY-formalism is expressed as contour integrals localized to the solutions of the scattering equations

$$S_a = 0, \qquad \text{where} \quad S_a = \sum_{b \neq a} \frac{s_{ab}}{z_{ab}}, \tag{1.1}$$

with $z_{ab} = z_a - z_b$ and z_a are auxiliary variables on the Riemann sphere. Unless otherwise specified, we let $a, b \in \{1, \ldots, n\}$. The momentum of the a^{th} external particle is k_a^μ and $s_{ab} = 2k_a \cdot k_b$ are the usual Mandelstam variables. The scattering equations are invariant under $\mathrm{PSL}(2, \mathbb{C})$ transformations of the variables,

$$z_a \to z'_a = \frac{Az_a + B}{Cz_a + D}, \qquad \text{where} \quad AD - BC = 1, \tag{1.2}$$

using overall momentum conservation, $\sum k_a = 0$, and the massless condition, $k_a^2 = 0$. This means that if z_a is a solution to eq. (1.1), then so is z'_a. Thus, only $(n-3)$ of the scattering equations are independent, which can be seen from the fact that

$$\sum_a S_a = \sum_a z_a S_a = \sum_a z_a^2 S_a = 0. \tag{1.3}$$

There is a redundancy in the integration variables which needs to be fixed, similar to how gauge redundancy is fixed. We choose three of the integration variables to be fixed, leaving $(n-3)$ unfixed variables, which are integrated over. Thus, the number of integration variables and the number of constraints from the scattering equations are equal, which fully localizes the integral to the solutions of the scattering equations. However, the number of independent solutions to the scattering equations is $(n-3)!$, and it becomes impractical to deal with them when n is not small. The computational cost becomes huge when the number of external particles increases. Integration rules have been developed to circumvent this problem, both at tree [4–9] and loop level [10], where no scattering equation has to be explicitly solved. A formal proof of the CHY-formalism was provided in ref. [11]. See also ref. [12].

Recently, one of us extended the scattering equation formalism to a double cover of the Riemann sphere (called the Λ-algorithm in refs. [13–16]). The auxiliary double-cover variables live in $\mathbb{CP}^2$, contrasted with the original auxiliary variables z_a, which live in $\mathbb{CP}^1$ in the standard CHY formulation. More precisely, we consider curves in $\mathbb{CP}^2$ defined by

$$C_a \equiv y_a^2 - \sigma_a^2 + \Lambda^2 = 0, \tag{1.4}$$

where Λ is a non-zero constant. This curve is invariant under a simultaneous scaling of the parameters y, σ, Λ. In the new double-cover formulation, the punctures on the Riemann sphere are given by the pair (σ_a, y_a). As eq. (1.4) is a quadratic equation, two branches

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develop. The value of y_a specifies which branch the solution is on. To make sure we pick up the puncture on the correct branch, the scattering equations have to be modified

$$\tilde{S}_a^\tau(\sigma, y) = \sum_{b\neq a} \frac{1}{2}\left(\frac{y_b}{y_a}+1\right)\frac{s_{ab}}{\sigma_{ab}}, \tag{1.5}$$

where $\sigma_{ab} = \sigma_a - \sigma_b$. The factor $\frac{1}{2}\left(\frac{y_b}{y_a}+1\right)$ projects out the solution where y_b approaches $-y_a$, and gives 1 when y_b approaches y_a. Another (equivalent) way of defining the double cover scattering equations is to postulate the map

$$S_a(z) = \sum_{a\neq b}\frac{s_{ab}}{z_{ab}} \to S_a^\tau(\sigma, y) = \sum_{a\neq b} s_{ab}\tau_{(a,b)}, \qquad \text{where} \quad \tau_{(a,b)} = \frac{1}{2\sigma_{ab}}\left(\frac{y_a+y_b+\sigma_{ab}}{y_a}\right). \tag{1.6}$$

It is easy to check that the two prescriptions for the double cover scattering equations are equivalent by using overall momentum conservation and the on-shell condition. The map $z_{ij} \to \tau^{-1}_{(i,j)}$ will be useful later when we define the double cover integrand. For a full formulation of the double-cover prescription, see ref. [13].

In the double cover prescription, three variables need to be fixed due to Möbius invariance. In addition, the integrand is invariant under a scale transformation. This gives an additional redundancy which needs to be fixed (as the integrand is $\mathrm{PSL}(2,\mathbb{C})$ and scale invariant, i.e. $\mathrm{GL}(2,\mathbb{C})$ invariant). Using the scale symmetry, we fix an extra puncture, and promote Λ to a variable and include a scale invariant measure $\frac{d\Lambda}{\Lambda}$. Using the global residue theorem, we can deform the integration contour to go around $\Lambda = 0$ instead of the solution to the scattering equation for the puncture fixed by the scale symmetry. This scattering equation is left free. Thus, in the double-cover prescription we gauge fix four points, three from the usual gauge fixing procedure, and one from the scale transformation.

The two sheets of the Riemann sphere are separated by a branch cut, and by integrating over Λ, lead to the factorization into two regular lower-point CHY amplitudes. This is the origin of the new factorization relations which we will discuss in the main part of this paper. By iteratively promoting the scattering amplitudes to the double-cover formulation, and using certain matrix identities, any n-point scattering amplitude for the non-linear sigma model can be fully factorized into off-shell three-point amplitudes.

This paper is organized as follows. In section 2 we formulate the non-linear sigma model amplitudes in the usual CHY formalism. In section 3 we introduce the double-cover prescription for effective field theories. In section 4 we describe the graphical representations for the scattering amplitudes in the double-cover formalism. In section 5 we list the double-cover integration rules. In section 6 we define the three-point functions which will serve as the building blocks for higher-point amplitudes. In sections 7 and 8 we present the new factorization formulas for the non-linear sigma model. In section 9 we present a novel recursion relation, which fully factorizes the non-linear sigma model amplitudes in terms of off-shell three-point amplitudes. This is one of the main results of the paper. Section 10 takes the soft limit of the non-linear sigma model amplitudes, and presents a new relation for NLSM $\oplus$ ϕ^3 amplitudes. In section 11 we apply the double-cover prescription

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to the special Galileon theory. We end with conclusions and outlook in section 12. The appendices A and B contain matrix identities and details of the six-point calculation.

2 CHY formalism

We briefly review the construction of non-linear sigma model (NLSM) scattering amplitudes in the CHY formalism to fix notation. The flavor-ordered partial U(N) amplitude for the non-linear sigma model in the scattering equation framework is defined by the integral

$$\mathcal{A}_n(\alpha) = \int \mathrm{d}\mu_n^{\mathrm{CHY}} (z_{pq} z_{qr} z_{rp})^2 H_n(\alpha), \tag{2.1}$$

$$\mathrm{d}\mu_n^{\mathrm{CHY}} = \prod_{a=1, a\neq p,q,r}^{n} \frac{\mathrm{d}z_a}{S_a}, \tag{2.2}$$

where a partial ordering is denoted by $(\alpha) = (\alpha_1, \ldots, \alpha_n)$. We have fixed the punctures $\{z_p, z_q, z_r\}$. The integrand is given by the Parke-Taylor factor $\mathrm{PT}(\alpha)$ and the reduced Pfaffian of the matrix A_n, $\mathrm{Pf}'\mathsf{A}_n$,

$$H_n(\alpha) = \mathrm{PT}(\alpha) \left(\mathrm{Pf}'\mathsf{A}_n\right)^2, \tag{2.3}$$

$$\mathrm{PT}(\alpha) = \frac{1}{z_{\alpha_1\alpha_2} z_{\alpha_2\alpha_3} \cdots z_{\alpha_n\alpha_1}}, \tag{2.4}$$

$$\left(\mathrm{Pf}'\mathsf{A}_n\right)^2 = \frac{(-1)^{i+j+l+m}}{z_{ij} z_{lm}} \mathrm{Pf}\left[(\mathsf{A}_n)_{ij}^{ij}\right] \times \mathrm{Pf}\left[(\mathsf{A}_n)_{lm}^{lm}\right]. \tag{2.5}$$

The matrix A_n is $n \times n$ and antisymmetric,

$$(\mathsf{A}_n)_{ab} = \begin{cases} \frac{s_{ab}}{z_{ab}} & \text{for} \quad a \neq b \\ 0 & \text{for} \quad a = b. \end{cases} \tag{2.6}$$

We will in general denote a reduced matrix by $(\mathsf{A}_n)_{j_1\ldots j_p}^{i_1\ldots i_p}$, where we have removed rows $\{i_1, \ldots, i_p\}$ and columns $\{j_1, \ldots, j_p\}$ from the matrix A_n. As an example, we can remove rows $\{i, j\}$ and columns $\{j, k\}$ from A_n in eq. (2.6), denoted by $(\mathsf{A}_n)_{jk}^{ij}$.

With the conventional choice $\{l, m\} = \{i, j\}$, the product of Pfaffians turns into a determinant

$$\left(\mathrm{Pf}'\mathsf{A}_n\right)^2 = -\mathrm{PT}(i, j) \det\left[(\mathsf{A}_n)_{ij}^{ij}\right]. \tag{2.7}$$

We will denote the amplitude with this choice by

$$A_n(\alpha) = -\int \mathrm{d}\mu_n^{\mathrm{CHY}} (z_{pq} z_{qr} z_{rp})^2 \,\mathrm{PT}(\alpha)\,\mathrm{PT}(i, j) \det\left[(\mathsf{A}_n)_{ij}^{ij}\right]. \tag{2.8}$$

We can make a different choice, specifically $\{l, m\} = \{j, k\}$. We will make use of the matrix identities

$$\mathrm{Pf}\left[(\mathsf{A}_n)_{ij}^{ij}\right] \times \mathrm{Pf}\left[(\mathsf{A}_n)_{jk}^{jk}\right] = \det\left[(\mathsf{A}_n)_{jk}^{ij}\right], \tag{2.9}$$

$$\det\left[(\mathsf{A}_n)_{jk}^{ij}\right] = 0 \qquad \text{if } n \text{ is odd.} \tag{2.10}$$

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Equation (2.10) depends on momentum conservation and the massless condition. A proof of the matrix identities in eqs. (2.9) and (2.10) is found in appendix A. The amplitude with this new choice is denoted by

$$A'_n(\alpha) = \int \mathrm{d}\mu_n^{\mathrm{CHY}} (z_{pq} z_{qr} z_{rp})^2 \, \mathrm{PT}(\alpha) \, \frac{(-1)^{i+k}}{z_{ij} z_{jk}} \det \left[(\mathsf{A}_n)^{ij}_{jk} \right] . \tag{2.11}$$

This definition differs from the conventional one, and will be of great practical use in the following [17]. It will often be useful to remove columns and rows from the set of fixed punctures. For the objects in eqs. (2.8) and (2.11), we will encode the information of which rows and columns are removed in the labeling of the partial ordering α. When removing columns and rows (i, j), we bold the corresponding elements in the partial ordering, i.e. $A_n(\dots, \boldsymbol{i}, \dots, \boldsymbol{j}, \dots)$. For the new prescription, the choice (ijk) is labeled by $A'_n(\dots, \boldsymbol{i}, \dots, \boldsymbol{j}, \dots, \boldsymbol{k}, \dots)$, where the set is chosen to be ordered as $i < j < k$. Unless otherwise specified, we assume the set of removed rows and columns are in the two or three first positions, i.e. $A_n = A_n(\boldsymbol{i}, \boldsymbol{j}, \dots)$ and $A'_n = A'_n(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \dots)$. In this case, we will suppress the bold notation. For an odd number of external particles n, $\det \left[(\mathsf{A}_n)^{ij}_{ij} \right] = \det \left[(\mathsf{A}_n)^{ij}_{jk} \right] = 0$, and the amplitudes vanish.

When evaluating the double cover amplitudes, it will be necessary to relax the requirement of masslessness, as the full amplitude is splits into off-shell lower-point amplitudes. The off-shell punctures are part of the set of fixed punctures. We will also use the object

$$A_n^{(ij)}(\alpha) = \int \mathrm{d}\mu_n^{\mathrm{CHY}} (z_{pq} z_{qr} z_{rp})^2 \, \mathrm{PT}(\alpha) \, \frac{(-1)^{i+j}}{z_{ij}} \det \left[(\mathsf{A}_n)^{i}_{j} \right] . \tag{2.12}$$

As the matrix A_n has co-rank 2 on the support of the massless condition and the scattering equations, $\{k_a^2 = 0, S_a = 0\}$, $A_n^{(ij)}(\alpha)$ vanishes trivially. However, when some of the particles are off-shell, $A_n^{(ij)}(\alpha)$ is non-zero in general. Similarly, the object $A'_n(\alpha)$ is non-zero for odd number of particles, if and only if some of the particles are off-shell.

3 Effective field theories in the double-cover prescription

In ref. [17], it was argued that the n-point NLSM scattering amplitude in the double-cover language is given by the integral

$$\mathcal{A}_n^{\mathrm{NLSM}}(\alpha) = \int_\Gamma \mathrm{d}\mu_n^\Lambda \, \frac{(-1)\Delta(pqr)\Delta(pqr|m)}{S_m^\tau} \, \mathcal{I}_n^{\mathrm{NLSM}}(\alpha), \tag{3.1}$$

$$\mathrm{d}\mu_n^\Lambda = \frac{1}{2^2} \frac{\mathrm{d}\Lambda}{\Lambda} \prod_{a=1}^{n} \frac{y_a \mathrm{d}y_a}{C_a} \prod_{d=1, d\neq p,q,r,m}^{n} \frac{\mathrm{d}\sigma_d}{S_d^\tau}, \tag{3.2}$$

$$\Delta(pqr) = \frac{1}{\tau_{(p,q)} \tau_{(q,r)} \tau_{(r,p)}}, \tag{3.3}$$

$$\Delta(pqr|m) = \sigma_p \Delta(qrm) - \sigma_q \Delta(rmp) + \sigma_r \Delta(mpq) - \sigma_m \Delta(pqr). \tag{3.4}$$

In this section we will include a superscript to denote the amplitudes. In the rest of the paper we keep this superscript implicit. When not otherwise specified, an amplitude without a superscript refers to an NLSM amplitude. The integration contour Γ is constrained

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by the $(2n-3)$ equations

$$\Lambda = 0, \qquad S_d^\tau(\sigma, y) = 0, \qquad C_a = 0, \tag{3.5}$$

for $d \neq \{p, q, r, m\}$ and $a = 1, \ldots, n$.

In a similar fashion, one can obtain the expressions for the NLSM $\oplus$ ϕ^3 and special Galileon amplitudes, i.e. for $A_n^{\text{NLSM}\oplus\phi^3}(\alpha||\beta)$ and A_n^{sGal}, by specifying the integrand. The integrands in the double-cover scattering equation framework for the NLSM, NLSM $\oplus$ ϕ^3 and special Galileon theory are given by the expressions

$$\mathcal{I}_n^{\text{NLSM}}(\alpha) = \text{PT}^\tau(\alpha) \times \mathbf{det}'\mathsf{A}_n^\Lambda\,, \tag{3.6}$$

$$\mathcal{I}_n^{\text{NLSM}\oplus\phi^3}(\alpha||\beta) = \text{PT}^\tau(\alpha)\left(\left[\prod_{a=1}^n \frac{(y\sigma)_a}{y_a}\right]\text{PT}^T(\beta)\,\text{det}\left[\mathsf{A}_n^\Lambda\right]_{\beta_1\ldots\beta_p}^{\beta_1\ldots\beta_p}\right), \tag{3.7}$$

$$\mathcal{I}_n^{\text{sGal}} = \mathbf{det}'\mathsf{A}_n^\Lambda \times \mathbf{det}'\mathsf{A}_n^\Lambda\,, \tag{3.8}$$

where $(y\sigma)_a \equiv y_a + \sigma_a$. The bold reduced determinant is defined as

$$\mathbf{det}'\mathsf{A}_n^\Lambda = \left[\prod_{a=1}^n \frac{(y\sigma)_a}{y_a}\right](-1)\,\text{PT}^T(i,j)\text{det}\left[\mathsf{A}_n^\Lambda\right]_{ij}^{ij} \tag{3.9}$$

$$= \left[\prod_{a=1}^n \frac{(y\sigma)_a}{y_a}\right](-1)^{i+k}T_{ij}T_{jk}\text{det}\left[\mathsf{A}_n^\Lambda\right]_{jk}^{ij}\,, \tag{3.10}$$

where the second equality is used to define the A' amplitude in the double cover language, similar to eq. (2.11). The Parke-Taylor factors and the kinematic matrix are defined by the following replacement

$$\mathsf{A}_n \to \mathsf{A}_n^\Lambda \qquad \text{for} \quad z_{ab} \to T_{ab}^{-1}, \tag{3.11}$$

$$\text{PT} \to \text{PT}^T \qquad \text{for} \quad z_{ab} \to T_{ab}^{-1}, \tag{3.12}$$

$$\text{PT} \to \text{PT}^\tau \qquad \text{for} \quad z_{ab} \to \tau_{(a,b)}^{-1}, \tag{3.13}$$

where $T_{ab}^{-1} = (y\sigma)_a - (y\sigma)_b$.

Notice that the generalization to theories such as sGal $\oplus$ NLSM^2 $\oplus$ ϕ^3 or Born-Infeld theory, among others, is straightforward [18–20].

3.1 The Π matrix

Most integrands in the CHY approach depend on the auxiliary variable z_i through the combination $z_{ij} = z_i - z_j$. As shown in eqs. (3.11) to (3.13) , we can construct the double cover integrand by replacing z_{ij} with T_{ij}^{-1} or $\tau_{(i,j)}^{-1}$.[1] This makes for an easy map between the traditional CHY approach and the new double cover method for most integrands.

However, the Π matrix, defined in refs. [18–20], has elements such as, $\frac{z_a\, k_a\cdot k_b}{z_{ab}}$, which so far have not been studied in the double cover framework. Explicitly, the $\Pi_{\beta_1,\beta_2,\ldots,\beta_m}$

[1] Of course, the measure is also redefined in the double cover prescription.

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matrix, defined in ref. [20], is

$$\Pi_{\beta_1,\ldots,\beta_m} = \left(\begin{array}{c|c|c|c} \begin{matrix} j\in\overline{\beta} \\ \mathsf{A}_{ij} \end{matrix} & \begin{matrix} b\in\{\beta_1,\ldots,\beta_m\} \\ \Pi_{ib} \end{matrix} & \begin{matrix} j\in\overline{\beta} \\ \mathsf{A}_{ij} \end{matrix} & \begin{matrix} b'\in\{\beta_1,\ldots,\beta_m\} \\ \Pi_{ib'} \end{matrix} \\ \hline \Pi_{aj} & \Pi_{ab} & \Pi_{aj} & \Pi_{ab'} \\ \hline \mathsf{A}_{ij} & \Pi_{ib} & 0 & \Pi_{ib'} \\ \hline \Pi_{a'j} & \Pi_{a'b} & \Pi_{a'j} & \Pi_{a'b'} \end{array}\right) \begin{matrix} i\in\overline{\beta} \\ a\in\{\beta_1,\ldots,\beta_m\} \\ i\in\overline{\beta} \\ a'\in\{\beta_1,\ldots,\beta_m\} \end{matrix} .$$

Here, the β_a's sets are such that $\beta_a \cap \beta_b = \emptyset$, $a \neq b$, and $\overline{\beta}$ is the complement, namely, $\overline{\beta} = \{1,2,\ldots,n\} \setminus \beta_1 \cup \beta_2 \cup \cdots \cup \beta_m$, where n is the total number of particles. The Π submatrices are given by the expressions

$$\Pi_{ib} = \sum_{c\in\beta_b} \frac{k_i\cdot k_c}{z_{ic}}, \qquad \Pi_{ib'} = \sum_{c\in\beta_b} \frac{z_c\, k_i\cdot k_c}{z_{ic}}, \qquad \Pi_{ab} = \sum_{\substack{c\in\beta_a \\ d\in\beta_b}} \frac{k_c\cdot k_d}{z_{cd}},$$
$$\Pi_{ab'} = \sum_{\substack{c\in\beta_a \\ d\in\beta_b}} \frac{z_d\, k_c\cdot k_d}{z_{cd}}, \quad \Pi_{a'b'} = \sum_{\substack{c\in\beta_a \\ d\in\beta_b}} \frac{z_c\, z_d\, k_c\cdot k_d}{z_{cd}}. \tag{3.14}$$

As shown in refs. [17, 21], to obtain the usual CHY matrices in the double-cover prescription we use the identification $\frac{1}{z_{ab}} \to T_{ab} = \frac{1}{(y_a+\sigma_a)-(y_b+\sigma_b)}$ (see the above section), which gives us the naive identification $z_a \to (y_a+\sigma_a)$. However, we need all elements of $\Pi_{\beta_1,\ldots,\beta_m}$ to transform in the same way under a global scaling $(y_1,\sigma_1,\ldots,y_n,\sigma_n,\Lambda) \to \rho\,(y_1,\sigma_1,\ldots,y_n,\sigma_n,\Lambda)$, $\rho\in\mathbb{C}^*$. We use the map[2] $z_a \to \frac{(y_a+\sigma_a)}{\Lambda}$. Thus, the Π matrix in the double-cover representation is given by the replacement,

$$\Pi^{\Lambda}_{\beta_1,\beta_2,\ldots,\beta_m} \equiv \Pi_{\beta_1,\beta_2,\ldots,\beta_m} \qquad \text{for} \quad \frac{1}{z_{ab}} \to T_{ab}, \quad z_a \to \frac{(y\sigma)_a}{\Lambda}. \tag{3.15}$$

The multi-trace amplitude for interactions among NLSM pions and bi-adjoint scalars is given by the integrand [20]

$$\mathcal{I}_n^{\text{NLSM}\oplus\text{BA}}(\alpha||\beta_1|\cdots|\beta_m) = \text{PT}^{\tau}(\alpha)\times\left(\left[\prod_{a=1}^{n}\frac{(y\sigma)_a}{y_a}\right]\times\text{PT}^{T}(\beta_1)\ldots\text{PT}^{T}(\beta_m)\times\text{Pf}'\left[\Pi^{\Lambda}_{\beta_1\ldots\beta_p}\right]\right).$$

The integrand is the defined using eqs. (3.12), (3.13) and (3.15). The reduced Pfaffian is defined as

$$\text{Pf}'\left[\Pi^{\Lambda}_{\beta_1\ldots\beta_p}\right] = \text{Pf}\left[(\Pi^{\Lambda}_{\beta_1\ldots\beta_p})^{ab'}_{ab'}\right]. \tag{3.16}$$

[2]This is in agreement with the single and double-cover equivalence given in ref. [13].

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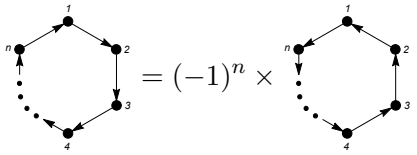
4 Graphical representation

The graphical representation for effective field theory amplitudes in the double-cover prescription is analogous to one presented in ref. [21]. The only difference is that we are going to work with determinants instead of Pfaffians. We will briefly review the graphical notation used in this paper.

First, the Parke-Taylor factor is drawn by a sequence of arrows joining vertices. The orientation of the arrow represents the ordering,

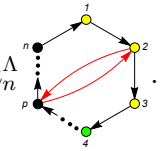
$$\mathrm{PT}^{\tau}(1,\ldots,n) = \quad = (-1)^n \times \quad = (-1)^n \times \mathrm{PT}^{\tau}(n,\ldots,1)\,. \tag{4.1}$$



To describe the half-integrand $(-1)\left[\prod_{a=1}^{n}\frac{(y\sigma)_a}{y_a}\right](T_{ij}T_{ji})\det[(\mathsf{A}_n^{\Lambda})^{ij}_{ij}]$, we recall how the Pfaffian in Yang-Mills theory was represented [21]. In YM, the half-integrand $(-1)^{i+j}\cdot\left[\prod_{a=1}^{n}\frac{(y\sigma)_a}{y_a}\right](T_{ij})\mathrm{Pf}[(\Psi_n^{\Lambda})^{ij}_{ij}]$ was represented by a red arrow from $i\to j$. We associate this red arrow with the factor T_{ij} of the reduced Pfaffian. In the case of NLSM, we draw two red arrows, $i\rightleftarrows j$, for the factor $T_{ij}T_{ji}$ of the reduced determinant. With the new definition of the NLSM integrand, $(-1)^{i+k}\left[\prod_{a=1}^{n}\frac{(y\sigma)_a}{y_a}\right]T_{ij}T_{jk}\det[(\mathsf{A}_n^{\Lambda})^{ij}_{jk}]$, we draw two red arrows, $i\to j\to k$.

If we choose to fix the punctures $(pqr|m) = (123|4)$ and reduce the determinant with $(i,j) = (2,p)$, we can graphically represent the NLSM amplitude $A_n(\alpha)$ by an *NLSM-graph*,

$$A_n(1,\mathbf{2},3,4,\ldots,\boldsymbol{p},\ldots,n) = \int d\mu_n^{\Lambda}$$



Recall that the removed columns and rows (i,j) are written in bold in the partial ordering. The notation for the fixed punctures by yellow, green and red vertices is the same as in ref. [21]. When all particles are on-shell, the expression is independent of the choice of fixed punctures and reduced determinant. However, as we shall see later, when we have off-shell particles, the expression depends on the choices.

Lastly, the following two properties

$$\begin{aligned} &A_n(1,\mathbf{2},3,4,\ldots,\boldsymbol{p},\ldots,n) = A_n(\mathrm{cyc}(1,\mathbf{2},3,4,\ldots,\boldsymbol{p},\ldots,n)),\\ &A_n(1,\mathbf{2},3,4,\ldots,\boldsymbol{p},\ldots,n) = (-1)^n\, A_n(n,\ldots,\boldsymbol{p},\ldots,4,3,\mathbf{2},1)\,, \end{aligned} \tag{4.2}$$

are satisfied even if some of the particles are off-shell. The graphical representation for other effective field theories are similar. Also, the double-cover representation reduces to the usual CHY representation when the green vertex is replaced by a black vertex.

5 The double-cover integration rules

We will formulate the double-cover integration rules, applicable for the effective field theory amplitudes for the NLSM and special Galileon theory (sGal). Generalizing the integration

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rules to other effective field theories is straightforward. The integration rules share a strong resemblance to the Yang-Mills integration rules given in ref. [21].

The integration of the double-cover variables y_a localizes the integrand to the curves $C_a = 0$, with the solutions $y_a = \pm\sqrt{\sigma_a^2 - \Lambda^2}$, $\forall\, a$. The double cover splits into an upper and a lower Riemann sheet, connected by a branch-cut, defined by the branch-points $-\Lambda$ and Λ. The punctures are distributed among the two sheets in all 2^n possible combinations.[3] When performing the integration of Λ, the two sheets factorize into two single covers connected by an off-shell propagator (the scattering equation S_m^τ in eq. (3.1) reduces to the off-shell propagator under the Λ integration). On each of the two lower-point single covers three punctures need to be fixed due to the $\mathrm{PSL}(2,\mathbb{C})$ redundancy. The branch-cut closes to a point when $\Lambda \to 0$, which becomes an off-shell particle. The corresponding puncture is fixed. In addition, two more punctures need to be fixed on each of the sheets. These fixed punctures must come from the fixed punctures in the original double cover (graphically represented by colored vertices, yellow or green). If there is not exactly two colored vertices on each of the new single covers, the configuration vanishes. We summarize this in the first integration rule [13, 21];

- *Rule-I. All configurations (or cuts) with fewer (or more) than two colored vertices (yellow or green) vanish trivially.*

The first integration rule, *Rule-I*, is general for any theory formulated in a double-cover language. In addition, we need to formulate supplementary integration rules specific to the NLSM and special Galileon amplitudes.

We start by determining how different parts of the integrand (and the measure) scale with Λ. Without loss of generality, consider a configuration where the punctures $\{\sigma_{p+1}, \ldots, \sigma_n, \sigma_1, \sigma_2\}$ are located on the upper sheet, and the punctures $\{\sigma_3, \sigma_4, \ldots, \sigma_p\}$ are located on the lower sheet. This configuration (or cut) will be graphically represented by a dashed red line, which separates the two sets. *Rule-I* forces two of the fixed punctures to be on the upper sheet, and the other two to be on the lower sheet. By expanding around $\Lambda = 0$, the measure and the Faddeev-Popov determinants become

$$\begin{aligned} d\mu_n^\Lambda\Big|_{3,\,4,\ldots,p}^{p+1,\ldots,1,2} &= \frac{d\Lambda}{\Lambda}\times\left[\frac{d\sigma_{p+1}}{S_{p+1}}\cdots\frac{d\sigma_n}{S_n}\right]\times\left[\frac{d\sigma_5}{S_5}\cdots\frac{d\sigma_p}{S_p}\right]+\mathcal{O}(\Lambda) \\ &= \frac{d\Lambda}{\Lambda}\times d\mu^{\mathrm{CHY}}_{n-(p-2)+1}\times d\mu^{\mathrm{CHY}}_{(p-2)+1}+\mathcal{O}(\Lambda), \end{aligned} \tag{5.1}$$

$$\frac{\Delta(123)\Delta(123|4)}{S_4^\tau}\Big|_{3,\,4,\ldots,p}^{p+1,\ldots,1,2} = \frac{2^5}{\Lambda^4}(\sigma_{12}\,\sigma_{2P_{3:p}}\,\sigma_{P_{3:p}1})^2\left[\frac{1}{s_{34\ldots p}}\right](\sigma_{P_{p+1:2}3}\,\sigma_{34}\,\sigma_{4P_{p+1:2}})^2+\mathcal{O}\left(\Lambda^{-2}\right), \tag{5.2}$$

where $P_{3:p}$ and $P_{p+1:2}$ denote the momentum of the off-shell punctures on the upper and lower sheets, respectively. Here, $P_{3:p} = k_3 + \cdots + k_p$, $P_{p+1:2} = k_{p+1} + \cdots + k_2$ and $s_{34\ldots p} = 2\sum_{i<j,i=3}^{p} k_i\cdot k_j$. For concreteness, we have fixed the punctures $(pqr|m) = (123|4)$.

[3] Only 2^{n-1} configurations are distinct, due to a $\mathbb{Z}_2$ symmetry.

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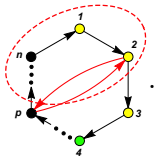
No. of cut arrows	Factor: $\mathrm{PT}^\tau(\alpha)$	Factor: $\mathbf{det}'(\mathsf{A}_n^\Lambda)$
0	Λ^0	Λ^0
1	—	Λ^2
2	Λ^2	Λ^2
3	—	—
4	Λ^4	—

Table 1. The table displays the dependence of Λ in the integrand factors when expanding around $\Lambda = 0$. Some entries are empty, meaning that they are impossible to achieve. E.g. the Parke-Taylor factor only appears when an even number of arrows are cut. This is because the PT factor forms a closed ring. Similarly, the reduced determinant enters with two arrows, so at most two arrows can be cut.

Graphically, this configuration is represented by

$$A_n(1,\mathbf{2},3,4,\ldots,\boldsymbol{p},\ldots,n)\Big|^{p+1,\ldots,1,2}_{3,4,\ldots,p} = \quad . \tag{5.3}$$



Notice how the measure and the Faddeev-Popov determinants scale with Λ at leading order,

$$d\mu_n^\Lambda \sim \frac{d\Lambda}{\Lambda}, \tag{5.4}$$

$$\frac{\Delta(123)\Delta(123|4)}{S_4^\tau} \sim \frac{1}{\Lambda^4}. \tag{5.5}$$

We also need to know how the Parke-Taylor factor and the reduced determinant scale with Λ. Table 1 shows how the integrand factors depend on Λ when expanded around $\Lambda = 0$. We see that how the integrand scales with Λ is very dependent on the number of cut arrows. For an NLSM amplitude, for each possible non-zero cut, we find that

$\mathrm{PT}^\tau(1,\ldots,n)\times\mathbf{det}'\mathsf{A}_n^\Lambda \sim \mathcal{O}(\Lambda^6)$, *The dashed red line cuts more than four arrows.*

$\mathrm{PT}^\tau(1,\ldots,n)\times\mathbf{det}'\mathsf{A}_n^\Lambda \sim \Lambda^4+\mathcal{O}(\Lambda^2)$, *The dashed red line cuts three or four arrows.*

$\mathrm{PT}^\tau(1,\ldots,n)\times\mathbf{det}'\mathsf{A}_n^\Lambda \sim \Lambda^2+\mathcal{O}(\Lambda^0)$, *The dashed red line cuts two arrows (singular cut).*

Similarly, for an sGal-graph, we find that

$\mathbf{det}'\mathsf{A}_n^\Lambda \times \mathbf{det}'\mathsf{A}_n^\Lambda \sim \Lambda^4 + \mathcal{O}(\Lambda^2)$, *The dashed red line cuts one or two arrows from each of the determinants.*

$\mathbf{det}'\mathsf{A}_n^\Lambda \times \mathbf{det}'\mathsf{A}_n^\Lambda \sim \Lambda^2 + \mathcal{O}(\Lambda^2)$, *The dashed red line cuts one or two arrows from a single the determinant (singular cut).*

$\mathbf{det}'\mathsf{A}_n^\Lambda \times \mathbf{det}'\mathsf{A}_n^\Lambda \sim \Lambda^0 + \mathcal{O}(\Lambda^2)$, *The dashed red line cuts no arrows (singular cut).*

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We combine this with eqs. (5.4) and (5.5). For an NLSM-graph, there is no residue when more than four arrows are cut, and the configuration vanishes. When three or four arrows are cut, the factor of $1/\Lambda^4$ from the Faddeev-Popov determinants is canceled by the integrand, and we have a simple pole in Λ. We can evaulate the contribution directly. However, when only two arrows are cut, we do not have a simple pole, and we need to expand beyond leading order. We call this configuration a *singular cut*. We summarize this in the second integration rule for an NLSM-graph;

- *Rule-II (NLSM-graph). If the dashed red line cuts fewer than three arrows over the NLSM-graph, the integrand must be expanded to next to leading order (singular cut). If the dashed red line cuts three or four arrows, the leading order expansion is sufficient. Otherwise, the cut is zero.*

We can perform a similar analysis for an sGal-graph. If one or two arrows from each of the determinants are cut, we have a simple pole and the contribution can be evaluated directly. Otherwise, the cut is singular and we need to expand beyond leading order. This produces the second integraion rule for an sGal-graph;

- *Rule-II (sGal-graph). If the dashed red line cuts at least one arrow from each of the determinants, the leading order expansion is sufficient. Otherwise, the integrand must be expanded to next to leading order.*

In ref. [13], this rule was called the Λ-theorem. In general, we want to avoid singular cuts. If the graph in question is regular (not singular), the following rule apply

- *Rule-IIIa (NLSM- and sGal-graphs). When the dashed red line cuts four arrows, the graph breaks into two smaller graphs (times a propagator). The off-shell puncture corresponds to a scalar particle.*

- *Rule-IIIb (NLSM- and sGal-graphs). If the dashed red line cuts three arrows in a graph, there is an off-shell vector field (gluon) propagating among the two resulting graphs. The two resulting graphs must be glued by the identity,* $\sum_M \epsilon^{M\,\mu}\,\epsilon^{M\,\nu} = \eta^{\mu\nu}$.

- *Rule-IIIc (sGal-graph). If the dashed red line cuts two arrows, there is an off-shell spin-2 field (graviton) propagating between the two resulting smaller graphs. The two sub-graphs are glued together by the identity* $\sum_M \epsilon^{M\,\mu\alpha}\epsilon^{M\,\nu\beta} = \eta^{\mu\nu}\eta^{\alpha\beta}$.

When there are off-shell gluons or gravitons connecting the sub-graphs, we must replace the corresponding off-shell momentum by a polarization vector, $P_i^\mu \to P_i^{M\,\mu} = \frac{1}{\sqrt{2}}\epsilon_i^{M\,\mu}$, in the reduced determinants [22].

Finally, we note that the integration rules are independent of the embedding,

- *Rule-IV. The number of intersection points among the dashed red-line and the arrows is given mod 2.*

We can always find an embedding where the dashed red line cuts any arrow zero or one time.

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6 Three-point functions

Before we look at examples, it is useful to compute the three-point amplitudes that will work as building blocks for higher-point amplitudes.

We are using the objects defined in eqs. (2.11) and (2.12). For the non-linear sigma model, the fundamental three-point functions are given by the expressions

$$A^{\phi^3}(P_a,P_b,P_c) = \quad = \int d\mu_3^{\rm CHY}(\sigma_{P_aP_b}\sigma_{P_bP_c}\sigma_{P_cP_a})^2\ {\rm PT}(P_a,P_b,P_c)^2 = 1\,, \tag{6.1}$$

$$A'_3(P_a,P_b,P_c) = \quad = \int d\mu_3^{\rm CHY}(\sigma_{P_aP_b}\sigma_{P_bP_c}\sigma_{P_cP_a})^2\,{\rm PT}(P_a,P_b,P_c)\,\frac{1}{\sigma_{P_aP_b}\,\sigma_{P_bP_c}}\,\frac{s_{P_cP_a}}{\sigma_{P_cP_a}}$$

$$= s_{P_cP_a}\,, \tag{6.2}$$

$$A_3^{(P_aP_b)}(P_a,P_b,P_c) = \quad = \int d\mu_3^{\rm CHY}(\sigma_{P_aP_b}\sigma_{P_bP_c}\sigma_{P_cP_a})^2\,{\rm PT}(P_a,P_b,P_c)$$

$$\times\frac{(-1)}{\sigma_{P_aP_b}}\det\begin{bmatrix}\frac{s_{P_bP_a}}{\sigma_{P_bP_a}} & \frac{s_{P_bP_c}}{\sigma_{P_bP_c}}\\ \frac{s_{P_cP_a}}{\sigma_{P_cP_a}} & 0\end{bmatrix} = s_{P_bP_c}\,s_{P_cP_a}\,, \tag{6.3}$$

where $P_a^\mu + P_b^\mu + P_c^\mu = 0$ and all particles could be off-shell, i.e. $P_i^2 \neq 0$. Using momentum conservation, we reformulate the expressions as

$$A'_3(P_a,P_b,P_c) = s_{P_cP_a} = -(P_a^2 - P_b^2 + P_c^2), \tag{6.4}$$

$$A_3^{(P_aP_b)}(P_a,P_b,P_c) = s_{P_bP_c}\,s_{P_cP_a} = (P_c^2 - P_a^2 + P_b^2)\times(P_a^2 - P_b^2 + P_c^2)$$
$$= A'_3(P_c,P_a,P_b)\times A'_3(P_a,P_b,P_c). \tag{6.5}$$

We see that the three-point functions in eqs. (6.4) and (6.5) vanish when the particles are on-shell.

7 Factorization relations

We will presents three different prescriptions for computing NLSM amplitudes. As we will see, they lead to three different factorization relations.

First, we start with the conventional NLSM prescription given in eq. (2.8) (in the double-cover language). It is useful to remember that for an odd number of external particles, the amplitude vanishes,

$$A_{2n+1}(1,\ldots,\boldsymbol{P_i},\ldots,\boldsymbol{P_j},\ldots,n) = 0. \tag{7.1}$$

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This relation holds even when the particles removed from the determinant by the choice (i, j) are off-shell, i.e. when $P_i^2 \neq 0$ and/or $P_j^2 \neq 0$.

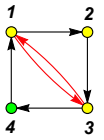
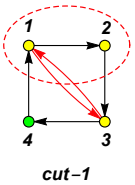
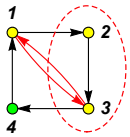
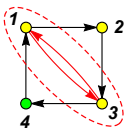
Secondly, we will use the alternative prescription given in eq. (2.11) with two different gauge fixing choices, resulting in two new factorization formulas. Parts of the results were reported by us in ref. [22].

In general, we denote the sum of cyclically-consecutive external momenta (modulo the total number of particles) by $P_{i:j} \equiv k_i + \cdots + k_j$. We also use the shorthand notation $P_{i,j} \equiv k_i + k_j$ for two (not necessarily consecutive) momenta. We also define the generalized Mandelstam variables $s_{i:i+j} \equiv s_{ii+1\ldots i+j}$ and $s_{i:i+j,L} \equiv s_{ii+1\ldots i+jL}$, with $s_{i_1\ldots i_p} \equiv \sum_{a\neq b,a,b=1}^{p} k_{i_a} \cdot k_{i_b}$.

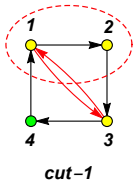
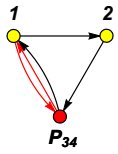
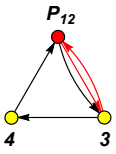
7.1 Four-point

7.1.1 The usual integrand prescription

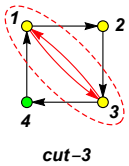
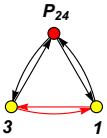
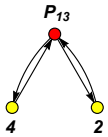
Let us start by considering the four-point amplitude, $A_4(1, 2, 3, 4)$. Without loss of generality, we choose the gauge fixing $(pqr|m) = (123|4)$. In order to avoid singular cuts (see section 5), we remove the columns and rows $(i, j) = (1, 3)$ for the determinant in eq. (2.8). For notational simplicity, we define $\mathbb{I}_n = (1, \ldots, n)$, $\mathbb{I}_n^{(ij)} = (1, \ldots, \boldsymbol{i}, \ldots, \boldsymbol{j}, \ldots, n)$, and $\mathbb{I}_n^{(ijk)} = (1, \ldots, \boldsymbol{i}, \ldots, \boldsymbol{j}, \ldots, \boldsymbol{k}, \ldots, n)$. Graphically, the amplitude factorizes into

$$A_4(\mathbb{I}_4^{(13)}) = \int d\mu_4^{\Lambda}$$  $$=$$  $$+$$  $$+$$  $$. \tag{7.2}$$

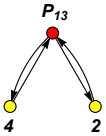
By applying *rule-III*, we can evaluate *cut-1*, finding

 $$=$$  $$\times \left(\frac{1}{s_{34}}\right) \times$$  $$= \frac{A_3(P_{34}, 1, 2) \times A_3(P_{12}, 3, 4)}{s_{34}} = 0, \tag{7.3}$$

where we have used eq. (7.1). *Cut-2* can be evaluated in a similar manner. Finally, it is straightforward to see that the last cut (*cut-3*) is broken into

 $$=$$  $$\times \left(\frac{1}{s_{24}}\right) \times$$  $$. \tag{7.4}$$

From the normalization of the three-point function in eq. (6.1), the first graph evaluates to (-1), while the second is (using *rule-III*)

 $$= \frac{(\sigma_{P_{13}2}\, \sigma_{24}\, \sigma_{4P_{13}})^2}{(\sigma_{P_{13}2}\, \sigma_{2P_{13}}) \times (\sigma_{P_{13}4} \sigma_{4P_{13}})} \times \det \begin{bmatrix} 0 & \frac{s_{24}}{\sigma_{24}} \\ \frac{s_{24}}{\sigma_{42}} & 0 \end{bmatrix} = s_{24}^2\,. \tag{7.5}$$

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We can also rewrite the cut using matrix relations defined in appendix A.2,

$$cut\text{-}\mathcal{3} = -\frac{A_3'(P_{13},2,4)A_3'(1,3,P_{24})}{s_{24}}. \tag{7.6}$$

By evaluating the cuts, we have that

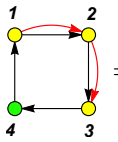
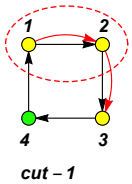
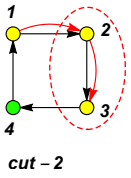
$$\begin{aligned} A_4(\mathbb{I}_4^{(13)}) &= \frac{A_3(P_{34},1,2)A_3(P_{12},3,4)}{s_{34}} + \frac{A_3(P_{23},1,4)A_3(3,P_{14},2)}{s_{23}} \\ &\quad - \frac{A_3'(P_{13},2,4)A_3'(1,3,P_{24})}{s_{24}} \\ &= -\frac{A_3'(P_{13},2,4)A_3'(1,3,P_{24})}{s_{24}} = -\frac{(-s_{13})\,(-s_{24})}{s_{24}} = -s_{13}. \end{aligned} \tag{7.7}$$

Here we have used eqs. (6.4) and (7.1) when evaluating the amplitude. Notice that the factorization channels with poles s_{34} and s_{23} vanish because they factorize into an odd NLSM amplitude, see eq. (7.1). The last contribution does not vanish, as it is not the usual NLSM prescription, but rather an off-shell amplitude with the new prescription given in eq. (2.11). Of course, the subamplitudes would vanish if all particles, including intermediate particles, were on-shell. In particular if P_{24} was on-shell (collinear limit). We can see this reflected by the answer, which would vanish in that case.

7.1.2 The new integrand prescription

In the previous section, we expressed the factorized non-linear sigma model amplitude with the usual prescription in terms of lower-point amplitudes with the new prescription. In this section we are going to do the calculations using the new prescription.

Let us consider the four-point amplitude, with gauge fixing $(pqr|m) = (123|4)$. In order to get a better understanding of the method, we are going to choose two different reduced determinants, i.e. we consider removing columns and rows such that $(ijk) = (123)$ in the first example, and $(ijk) = (134)$ in the second example. In the first example, we have the graphical representation

$A_4'(\mathbb{I}_4) = \int d\mu_4^{\Lambda}$  $=$  $+$  . (7.8)

The graphs can be evaluated as

$$A_4'\left(\mathbb{I}_4\right) = \sum_M \left[\frac{A_3'(1,2,P_{34}^M)A_3^{(P_{12}3)}(P_{12}^M,3,4)}{s_{34}} + \frac{A_3^{(1P_{23})}(1,P_{23}^M,4)A_3'(P_{41}^M,2,3)}{s_{41}} \right]. \tag{7.9}$$

We see that all factorization contributions are glued together by an off-shell vector field (off-shell gluon). The notation P_i^M means the replacement $P_i^\mu \to \frac{1}{\sqrt{2}}\epsilon_i^{M\,\mu}$ in the A_n matrix. Also, the gluing relation is

$$\sum_M \epsilon_i^{M\,\mu}\epsilon_j^{M\,\nu} = \eta^{\mu\nu}. \tag{7.10}$$

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Explicitly, the two factorization contributions become

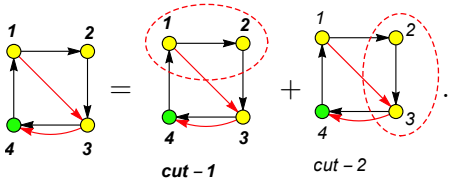
$$\sum_M \frac{A'_3(1,2,P_{34}^M)A_3^{(P_{12}3)}(P_{12}^M,3,4)}{s_{34}} = \sum_M \frac{\left(\sqrt{2}\epsilon_{34}^M\cdot k_1\right)\times s_{34}\left(\sqrt{2}\epsilon_{12}^M\cdot k_4\right)}{s_{34}} = \frac{s_{14}s_{34}}{s_{34}} = s_{14}, \tag{7.11}$$

and

$$\sum_M \frac{A_3^{(1P_{23})}(1,P_{23}^M,4)A'_3(P_{41}^M,2,3)}{s_{23}} = \sum_M \frac{\left(\sqrt{2}\epsilon_{23}^M\cdot k_4\right)s_{41}\times\left(\sqrt{2}\epsilon_{41}^M\cdot k_3\right)}{s_{23}} = \frac{s_{14}s_{34}}{s_{23}} = s_{12}. \tag{7.12}$$

As a second example, consider

$$A'_4(\mathbb{I}_4^{(134)}) = \int d\mu_4^\Lambda$$



. (7.13)

The graphs evaluate to

$$A'_4(\mathbb{I}_4^{(134)}) = \sum_M \frac{A_3^{(1P_{34})}(1,2,P_{34}^M)A'_3(P_{12}^M,3,4)}{s_{34}} + \frac{A'_3(1,P_{23},4)A_3(3,P_{41},2)}{s_{23}}. \tag{7.14}$$

Notice that only one of the factorization contributions (*cut-1*) is glued together by an off-shell gluon, while the second contribution (*cut-2*) is a purely scalar contribution. Evaluating the contributions, we find that

$$\sum_M \frac{A_3^{(1P_{34})}(1,2,P_{34}^M)A'_3(P_{12}^M,3,4)}{s_{34}} = \sum_M \frac{-\left(\sqrt{2}\epsilon_{34}^M\cdot k_2\right)s_{12}\times\left(\sqrt{2}\epsilon_{12}^M\cdot k_4\right)}{s_{34}} = -\frac{s_{12}s_{24}}{s_{34}} = -s_{13}, \tag{7.15}$$

and

$$\frac{A'_3(1,P_{23},4)A_3(3,P_{41},2)}{s_{23}} = \frac{P_{23}^2\times 0}{s_{23}} = 0. \tag{7.16}$$

The scalar contribution vanishes, as an odd amplitude in the usual prescription vanishes, see eq. (7.1).

Summing the contributions, we obtain

$$A'_4(\mathbb{I}_4^{(123)}) = s_{14}+s_{12} = -s_{13}, \tag{7.17}$$

$$A'_4(\mathbb{I}_4^{(134)}) = -s_{13}+0 = -s_{13}. \tag{7.18}$$

This agrees with eq. (7.7).

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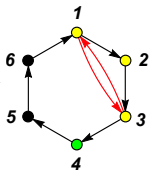
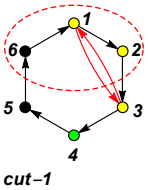
7.2 Six-point

Next, we compute the six-point amplitude using the double-cover formalism. We stick to the gauge fixing $(pqr|m) = (123|4)$, and to removing the columns and rows $(i, j) = (1, 3)$. Graphically, the amplitude factorizes into

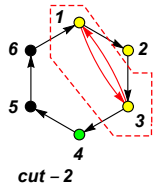
$$A_6(\mathbb{I}_6^{(13)}) = \int d\mu_6^{\Lambda}$$



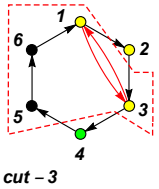
$$=$$



$$+$$

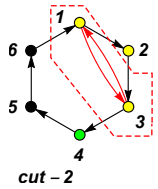


$$+$$

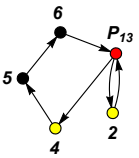


$$. \quad (7.19)$$

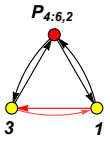
We have omitted some factorizations, which evaluate to zero by analogy to the four-point case. Note that, the *cut-1* is straightforward to evaluate, as it factorizes into lower-point NLSM amplitudes. However, *cut-2* and *cut-3* do not have straightforward interpretations (which is why they sometimes are referred to as *strange-cuts*). Take *cut-2* as an example, it graphically takes the form



$$= \int d\mu_5^{\text{CHY}}$$



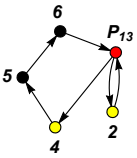
$$\times \left(\frac{1}{s_{4:6,2}}\right) \times$$



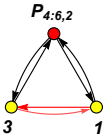
$$. \quad (7.20)$$

The first graph looks non-simple to be computed since there is no way to avoid the *singular cuts*. Nevertheless, such as in Yang-Mills theory, ref. [21], this strange-cut can be rewritten in the following way

$$\int d\mu_5^{\text{CHY}}$$



$$\times$$



$$= (-1)\, A_5'(P_{13}, 2, 4, 5, 6) \times A_3'(1, 3, P_{4:6,2}), \quad (7.21)$$

which comes from the matrix identities given in appendix A.2. We can do a similar rewriting for *cut-3*. The full calculation is presented in appendix B.3.

Putting it all together, the six-point amplitude factorizes as

$$\begin{aligned} A_6(\mathbb{I}_6^{(13)}) &= \frac{A_4(\mathbf{1}, 2, \boldsymbol{P_{3:5}}, 6) A_4(P_{6:2}, 3, 4, 5)}{s_{3:5}} - \frac{A_5'(P_{13}, 2, 4, 5, 6) A_3'(1, 3, P_{4:6,2})}{s_{13}} \\ &\quad - \frac{A_3'(P_{5:1,3}, 2, 4) A_5'(1, 3, P_{24}, 5, 6)}{s_{24}} \\ &= \frac{s_{26} s_{35}}{s_{3:5}} + s_{13}\left[\frac{s_{46}}{s_{4:6}} + \frac{s_{26}+s_{46}}{s_{56P_{13}}}\right] + s_{24}\left[\frac{s_{26}+s_{46}}{s_{56P_{24}}} + \frac{s_{26}+s_{36}+s_{46}}{s_{5:1}}\right]. \end{aligned} \quad (7.22)$$

By using momentum conservation, all unphysical poles cancel, and we match with the known result

$$\begin{aligned} A_6\left(\mathbb{I}_6\right) &= \frac{(s_{12}+s_{23})(s_{45}+s_{56})}{s_{123}} + \frac{(s_{23}+s_{34})(s_{56}+s_{61})}{s_{234}} + \frac{(s_{34}+s_{45})(s_{56}+s_{61})}{s_{345}} \\ &\quad - (s_{12}+s_{23}+s_{34}+s_{45}+s_{56}+s_{61}). \end{aligned} \quad (7.23)$$

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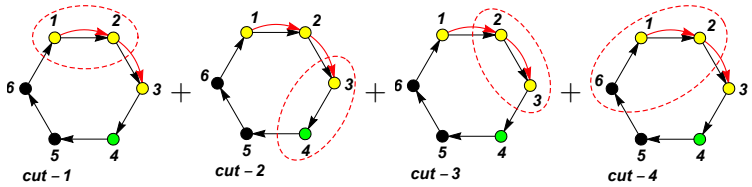
The six-point amplitude can also be computed using the new prescription. The first example with the choice $(ijk) = (123)$ gives, graphically,

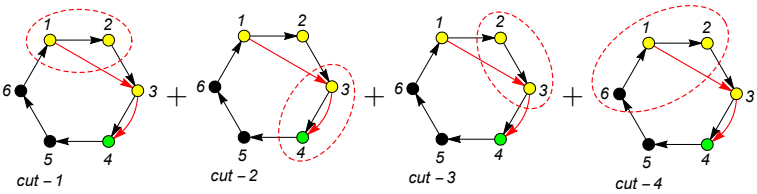
$$A'_6(\mathbb{I}_6^{(123)}) =$$ 

$$. \tag{7.24}$$

We have carried out the full calculation in appendix B.1. The contributions unambiguously evaluate to

$$\begin{aligned} A'_6(\mathbb{I}_6^{(123)}) = & \\ \sum_M \Bigg[& \frac{A'_3\left(1,2,P^M_{3:6}\right) A_5^{(P_{12}3)}(P^M_{12},3,4,5,6)}{s_{3:6}} + \frac{A'_5(1,2,P^M_{34},5,6) A_3^{(P_{5:2}3)}(P^M_{5:2},3,4)}{s_{34}} \\ & + \frac{A'_3\left(P^M_{4:1},2,3\right) A_5^{(1P_{23})}(1,P^M_{23},4,5,6)}{s_{4:1}} + \frac{A'_4\left(1,2,P^M_{3:5},6\right) A_4^{(P_{6:2}3)}\left(P^M_{6:2},3,4,5\right)}{s_{3:5}} \Bigg]. \end{aligned} \tag{7.25}$$

Graphically, the second example, with the choice $(ijk) = (134)$, is

$$A'_6(\mathbb{I}_6^{(134)}) =$$ 

$$, \tag{7.26}$$

which becomes (see appendix B.2 to follow the full computation)

$$\begin{aligned} A'_6(\mathbb{I}_6^{(134)}) = & \\ \sum_M \Bigg[& \frac{A_3^{(1P_{3:6})}\left(1,2,P^M_{3:6}\right) A'_5(P^M_{12},3,4,5,6)}{s_{3:6}} + \frac{A_5^{(1P_{34})}(1,2,P^M_{34},5,6) A'_3(P^M_{5:2},3,4)}{s_{34}} \\ & + \frac{A_4^{(1P_{3:5})}\left(1,2,P^M_{3:5},6\right) A'_4\left(P^M_{6:2},3,4,5\right)}{s_{3:5}} \Bigg] + \frac{A_3\left(3,P_{4:1},2\right) A'_5(1,P_{23},4,5,6)}{s_{4:1}}. \end{aligned} \tag{7.27}$$

Notice that the last contribution (*cut-3*) evaluates to zero. We can check that both examples with the new integrand prescription reproduce the correct result. The full six-point calculation for both choices of gauge fixing is presented in appendix B. Notice that in the first example, all factorization contributions are glued together with off-shell gluons, while in the second example, three contributions involve off-shell gluons, and one contribution is purely in terms of scalar particles.

So far we have seen three different kinds of factorization relations. The first kind, presented in eqs. (7.7) and (7.22), all particles were scalar. In the second case, given by eqs. (7.9) and (7.25), the intermediate particles were vector fields (off-shell gluons). Finally,

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in the last case, eqs. (7.14) and (7.27), the factorization relation involved both intermediate scalar and vector fields.[4]

7.3 Longitudinal contribution

As the non-linear sigma model is a scalar theory, it is an interesting proposition to only consider longitudinal contributions. An off-shell vector field can be decomposed in terms of transverse and longitudinal degrees of freedom. Let us consider only including the longitudinal degrees of freedom.

Practically, this means that instead of using the relation in eq. (7.10), we keep only the longitudinal sector,

$$\sum_L \epsilon_i^{L\,\mu}\epsilon_j^{L\,\nu} = \frac{k_i^\mu k_j^\nu}{k_i\cdot k_j} = \overline{k}_i^\mu k_j^\nu, \quad \text{with,}\;\; k_i^\mu = -k_j^\mu, \;\; \overline{k}_i^\mu = -\left(\frac{k_i^\mu}{k_i^2}\right). \tag{7.28}$$

Here we label the polarization vectors by a superscript L instead of M when keeping only longitudinal degrees of freedom.

In the four-point example, we have that

$$\begin{aligned}\sum_L \left[\frac{A_3'(1,2,P_{34}^L)A_3^{(P_{12}3)}(P_{12}^L,3,4)}{s_{34}} + \frac{A_3^{(1P_{23})}(1,P_{23}^L,4)A_3'(P_{41}^L,2,3)}{s_{23}}\right] \\ = -\frac{1}{2}\left[\frac{s_{12}^2}{s_{12}} + \frac{s_{14}^2}{s_{14}}\right] = \frac{s_{13}}{2} = -\frac{1}{2}A_4(\mathbb{I}_4)\end{aligned} \tag{7.29}$$

and

$$\begin{aligned}\sum_L \frac{A_3^{(1P_{34})}(1,2,P_{34}^L)A_3'(P_{12}^L,3,4)}{s_{34}} + \frac{A_3'(1,P_{23},4)A_3(3,P_{41},2)}{s_{23}} \\ = \frac{1}{2}\left[\frac{s_{12}^2}{s_{12}} + \frac{0}{s_{14}}\right] = \frac{s_{12}}{2} \neq \rho A_4(\mathbb{I}_4)\end{aligned} \tag{7.30}$$

where is ρ is a real constant. The sum of longitudinal contributions in eq. (7.29) is proportional to the correct answer, while the sum of longitudinal contributions in eq. (7.30) is not.

Applying the same ideas to the six-point amplitude in eq. (7.25), we have that

$$\begin{aligned}\sum_L \Bigg[&\frac{A_3'\left(1,2,P_{3:6}^L\right)A_5^{(P_{12}3)}(P_{12}^L,3,4,5,6)}{s_{3:6}} + \frac{A_5'(1,2,P_{34}^L,5,6)A_3^{(P_{5:2}3)}(P_{5:2}^L,3,4)}{s_{34}} \\ &+ \frac{A_3'\left(P_{4:1}^L,2,3\right)A_5^{(1P_{23})}(1,P_{23}^L,4,5,6)}{s_{4:1}} + (-1)\frac{A_4'\left(1,2,P_{3:5}^L,6\right)A_4^{(P_{6:2}3)}\left(P_{6:2}^L,3,4,5\right)}{s_{3:5}}\Bigg] \\ &= -\frac{1}{2}A_6(\mathbb{I}_6).\end{aligned} \tag{7.31}$$

[4] Although in this case, the factorization contribution where the propagated particle is a scalar field vanishes, it is simple to find an example where this does not happen. For instance, let us choose the gauge, $(pqr|m) = (134|6)$, and the reduced A_n matrix with $(ijk) = (146)$. It is not hard to check that for this gauge fixing the amplitude, $A_6'(\mathbb{I}_6^{(146)})$, has the two types of factorization contributions which are non-zero.

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Notice that the relative sign of the contribution from even subamplitudes (physical pole) was flipped in order to reproduce the correct amplitude.[5] In the four-point example, all subamplitudes are odd, and no relative sign flip is needed. All the longitudinal contributions are computed in appendix B.4.

Now, let us focus on the factorization relation given in eq. (7.27) and its longitudinal contributions

$$\sum_L \Bigg[(-1)^{i_1}\frac{A_3^{(1P_{3:6})}\left(1,2,P^L_{3:6}\right)A'_5(P^L_{12},3,4,5,6)}{s_{3:6}}+(-1)^{i_2}\frac{A_5^{(1P_{34})}(1,2,P^L_{34},5,6)A'_3(P^L_{5:2},3,4)}{s_{34}}$$
$$+(-1)^{i_3}\frac{A_4^{(1P_{3:5})}\left(1,2,P^L_{3:5},6\right)A'_4\left(P^L_{6:2},3,4,5\right)}{s_{3:5}}\Bigg]+\frac{A_3\left(3,P_{4:1},2\right)A'_5(1,P_{23},4,5,6)}{s_{4:1}}$$
$$\neq \rho A_6(\mathbb{I}_6), \tag{7.32}$$

where the non-equality is preserved for all $2^3=8$ possible combinations of relative signs, i.e. $(i_1,i_2,i_3)\in\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$. Thus, like the four-point example, the amplitude with both off-shell gluons and scalars does not reproduce the full answer when only longitudinal contributions are kept. Again, the longitudinal contributions are presented in appendix B.4.

In summary, we have obtained examples of three different factorization relations, involving only intermediate scalars, off-shell gluons, or both scalars and off-shell gluons, respectively. In the case where we have only off-shell gluons, we are able to reproduce the full answer by only keeping the longitudinal degrees of freedom (with a relative sign flip between even and odd factorization contributions).

8 General factorization relations

The factorization relations from the previous section can be generalized. In this section, we present three different factorization formulas. One formula is given in terms of exchange of off-shell vector fields, while the other two formulas are given in terms of purely scalar fields.

First, let us consider the case, $A_{2n}(\mathbb{I}^{(13)}_{2n})$. Thus, as in the section 7.1.1, we choose the gauge fixing $(pqr|m)=(123|4)$ and the reduced matrix with $(ij)=(13)$, namely $[\mathsf{A}_{2n}]^{13}_{31}$. Applying the integration rules, the amplitude becomes

$$A_{2n}(\mathbb{I}^{(13)}_{2n})=\sum_{i=3}^{n}\frac{A'_{2(n-i+2)}\left(1,2,P_{3:2i-1},2i,\ldots,2n\right)\times A'_{2(i-1)}\left(P_{2i:2},3,4,\ldots,2i-1\right)}{s_{3:2i-1}}+$$
$$(-1)\sum_{i=3}^{n+1}\frac{A'_{2(n-i+2)+1}\left(1,3,P_{4:2i-2,2},2i-1,\ldots,2n\right)\times A'_{2(i-1)-1}\left(P_{2i-1:1,3},2,4,\ldots,2i-2\right)}{s_{4:2i-2,2}}. \tag{8.1}$$

[5] We have tested all possible sign combinations, and this is the only one which is proportional to the correct amplitude.

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This formula has been check up to ten points. In order to obtain the this relation, we used the matrix identities formulated in appendix A.2. In the first line, we used that

$$\begin{aligned} A_{2i}\left(\ldots,\boldsymbol{P_p},\ldots,\boldsymbol{P_q},\ldots,P_r,\ldots\right) &= A_{2i}\left(\ldots,\boldsymbol{P_p},\ldots,P_q,\ldots,\boldsymbol{P_r},\ldots\right) \\ &= A_{2i}\left(\ldots,P_p,\ldots,\boldsymbol{P_q},\ldots,\boldsymbol{P_r},\ldots\right) \\ &= A'_{2i}\left(\ldots,\boldsymbol{P_p},\ldots,\boldsymbol{P_q},\ldots,\boldsymbol{P_r},\ldots\right). \end{aligned} \tag{8.2}$$

For the second line, we used properties *I* and *III* in appendix A.2.

Thus, as the formula obtained in eq. (8.1), our second factorization relation, that was already presented in ref. [22], is supported on the double-cover formalism. In order to generalize the eqs. (7.9) and (7.25), we choose the same gauge fixing, $(pqr|m) = (123|4)$, and the reduced matrix with, $(ijk) = (123)$, (i.e. $[\mathsf{A}_{2n}]^{12}_{23}$). By the integration rules formulated in section 5, it is straightforward to see the amplitude turns into

$$\begin{aligned} A'_{2n}\left(\mathbb{I}_{2n}\right) = \sum_{M}\Bigg[&\sum_{i=3}^{n}\frac{A'_{2(n-i+2)}\left(1,2,P^{M}_{3:2i-1},2i,\ldots,2n\right)\times A^{(P_{2i:2}3)}_{2(i-1)}\left(P^{M}_{2i:2},3,4,\ldots,2i-1\right)}{s_{3:2i-1}} \\ &+\sum_{i=3}^{n+1}\frac{A'_{2(n-i+2)+1}\left(1,2,P^{M}_{3:2i-2},2i-1,\ldots,2n\right)\times A^{(P_{2i-1:2}3)}_{2(i-1)-1}\left(P^{M}_{2i-1:2},3,4,\ldots,2i-2\right)}{s_{3:2i-2}} \\ &+\frac{A'_{3}\left(P^{M}_{4:1},2,3\right)\times A^{(1P_{23})}_{2n-1}\left(1,P^{M}_{23},4,\ldots,2n\right)}{s_{4:1}}\Bigg], \end{aligned} \tag{8.3}$$

where we use eq. (7.10). This second general formula has been verified up to ten points.

On the other hand, from the results obtained in the eqs. (7.29) and (7.31) for four and six points, respectively, we can generalize the idea presented in section 7.3 to higher number of points. Therefore, by considering just the longitudinal degrees of freedom in eq. (8.3), we conjecture the following factorization formula [22],

$$\begin{aligned} A'_{2n}(\mathbb{I}_{2n}) = 2\sum_{L}\Bigg[&\sum_{i=3}^{n}\frac{A'_{2(n-i+2)}\left(1,2,P^{L}_{3:2i-1},2i,\ldots,2n\right)\times A^{(P_{2i:2}3)}_{2(i-1)}\left(P^{L}_{2i:2},3,4,\ldots,2i-1\right)}{s_{3:2i-1}} \\ &+(-1)\sum_{i=3}^{n+1}\frac{A'_{2(n-i+2)+1}\left(1,2,P^{L}_{3:2i-2},2i-1,\ldots,2n\right)\times A^{(P_{2i-1:2}3)}_{2(i-1)-1}\left(P^{L}_{2i-1:2},3,4,\ldots,2i-2\right)}{s_{3:2i-2}} \\ &+(-1)\frac{A'_{3}\left(P^{L}_{4:1},2,3\right)\times A^{(1P_{23})}_{2n-1}\left(1,P^{L}_{23},4,\ldots,2n\right)}{s_{4:1}}\Bigg], \end{aligned} \tag{8.4}$$

where we use eq. (7.28). Finally, by applying the identities

$$\begin{aligned} A^{(P_pP_q)}_{2i}\left(\ldots,P_p,\ldots,P_q,\ldots,P_r,\ldots\right) &= A^{(P_qP_r)}_{2i}\left(\ldots,P_p,\ldots,P_q,\ldots,P_r,\ldots\right) \\ &= -(P_p^2+P_q^2+P_r^2)\times A'_{2i}\left(\ldots,\boldsymbol{P_p},\ldots,\boldsymbol{P_q},\ldots,\boldsymbol{P_r},\ldots\right), \\ A^{(P_pP_q)}_{2i+1}\left(\ldots,P_p,\ldots,P_q,\ldots,P_r,\ldots\right) &= A^{(P_qP_r)}_{2i+1}\left(\ldots,P_p,\ldots,P_q,\ldots,P_r,\ldots\right) \\ &= (P_p^2-P_q^2-P_r^2)\times A'_{2i+1}\left(\ldots,\boldsymbol{P_p},\ldots,\boldsymbol{P_q},\ldots,\boldsymbol{P_r},\ldots\right), \end{aligned} \tag{8.5}$$

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which are a consequence from the properties in appendix A.2, it is straightforward to see the eq. (8.4) becomes

$$A'_{2n}(\mathbb{I}_{2n}) = \sum_{i=3}^{n} \frac{A'_{2(n-i+2)}\big(1,2,P_{3:2i-1},2i,\ldots,2n\big)\times A'_{2(i-1)}\big(P_{2i:2},3,4,\ldots,2i{-}1\big)}{s_{3:2i-1}}$$
$$+\sum_{i=3}^{n+1} \frac{A'_{2(n-i+2)+1}\big(1,2,P_{3:2i-2},2i{-}1,\ldots,2n\big)\times A'_{2(i-1)-1}\big(P_{2i-1:2},3,4,\ldots,2i{-}2\big)}{s_{3:2i-2}}$$
$$+(-1)\frac{A'_3\big(P_{4:1},2,3\big)\times A'_{2n-1}\big(1,P_{23},4,\ldots,2n\big)}{s_{4:1}}. \tag{8.6}$$

This is our third general factorization formula.

8.1 A new relationship for the boundary terms

As we argued in ref. [22], the amplitudes with an odd number of particles, i.e. amplitudes of the form $A'_{2m+1}(\ldots,\boldsymbol{P}_a,\ldots)$ (odd amplitude), are proportional to P_a^2 since that them must vanish when all particles are on-shell. Thus, the poles given by the odd contributions, namely expressions of the form $\frac{A'_{2m+1}(\ldots,\boldsymbol{P}_a,\ldots)\times A'_{2k+1}(\ldots,\boldsymbol{P}_b,\ldots)}{2\,P_a\cdot P_b}$, are spurious and, therefore, those terms are on the boundary of any usual BCFW deformation [23]. In particular, under the BCFW deformation,

$$k_2^\mu(z) = k_2^\mu + z\,q^\mu\,, \qquad k_3^\mu(z) = k_3^\mu - z\,q^\mu\,, \quad \text{with} \;\; q^2 = 0, \tag{8.7}$$

all even contributions (physical poles), which are given by the sum

$$\sum_{i=3}^{n} \frac{A'_{2(n-i+2)}\big(1,2,P_{3:2i-1},2i,\ldots,2n\big)\times A'_{2(i-1)}\big(P_{2i:2},3,4,\ldots,2i-1\big)}{P^2_{3:2i-1}(z)} \tag{8.8}$$

in eqs. (8.1) and (8.6), are localized over the z-plane at, $P^2_{3:2i-1}(z)=0$. Thus, by the above discussion, all odd contributions in eqs. (8.1) and (8.6) are localized at the point $z=\infty$ on the z-plane and, hence, we call those odd amplitudes the boundary terms.

Now, clearly, by comparing the factorization relations obtained in eqs. (8.1) and (8.6), this is straightforward to see that one arrives to the identity

$$\sum_{i=3}^{n+1} \frac{A'_{2(n-i+2)+1}\big(1,2,P_{3:2i-2},2i{-}1,\ldots,2n\big)\times A'_{2(i-1)-1}\big(P_{2i-1:2},3,4,\ldots,2i{-}2\big)}{s_{3:2i-2}} + (2 \leftrightarrow 3)$$
$$= \frac{A'_3\big(P_{4:1},2,3\big)\times A'_{2n-1}\big(1,P_{23},4,\ldots,2n\big)}{s_{4:1}}\,, \tag{8.9}$$

which lies on the boundary of any usual BCFW deformation. We have checked this identity up to $n=10$.

9 A novel recursion relation

In this section, we are going to present a new recursion relationship, which can be used to write down any NLSM amplitude in terms of the three-point building-block, $A'_3(P_a,P_b,P_c)= -(P_a^2-P_b^2+P_c^3)$, given in eq. (6.2).

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Previously, in eq. (8.4), we arrived at an unexpected factorization expansion, which, although it emerged accidentally from the integration rules, a formal proof is yet unknown.[6] Thus, since applying the integration rules is an iterative process, we would like to know if the relationship in eq. (8.4) could be extended to off-shell amplitudes (both for an even and odd number of particles). Here, we are going to show how to do that.

First, consider the four-point computation, $A'_4(P_1, P_2, P_3, 4)$, where the particles, $\{P_1, P_2, P_3\}$, can be off-shell. By the integration rules, we obtain the same decomposition as in eq. (7.9),

$$A'_4\left(P_1, P_2, P_3, 4\right) = \tag{9.1}$$
$$\sum_M \left[\frac{A'_3(P_1, P_2, P_{34}^M) A_3^{(P_{12}P_3)}(P_{12}^M, P_3, 4)}{s_{P_3P_4}} + \frac{A_3^{(P_1P_{23})}(P_1, P_{23}^M, 4) A'_3(P_{41}^M, P_2, P_3)}{s_{P_4P_1}}\right] = -s_{4P_2}.$$

Now, by using the longitudinal gluing relation given in eq. (7.28), i.e. $\sum_L \epsilon_{34}^{\mu\, L}\epsilon_{12}^{\nu\, L} = \overline{P}_{34}^{\mu} P_{12}^{\nu}$ and $\sum_L \epsilon_{23}^{\mu\, L}\epsilon_{41}^{\nu\, L} = P_{23}^{\mu}\overline{P}_{41}^{\nu}$, over the above factorized amplitude, one arrives at

$$(-2)\sum_L \left[\frac{A'_3(P_1, P_2, P_{34}^L) A_3^{(P_{12}P_3)}(P_{12}^L, P_3, 4)}{s_{P_3P_4}} + \frac{A_3^{(P_1P_{23})}(P_1, P_{23}^L, 4) A'_3(P_{41}^L, P_2, P_3)}{s_{P_4P_1}}\right]$$
$$= \frac{-(P_1^2 - P_2^2 + P_{34}^2)\, s_{4P_{12}}}{P_{34}^2} + \frac{-(P_{41}^2 - P_2^2 + P_3^2)\, s_{4P_{23}}}{P_{41}^2}. \tag{9.2}$$

Clearly, since $\{P_1, P_2, P_3\}$ are off-shell, the results found in eqs. (9.1) and (9.2) do not match. However, there is a simple way to make them coincide. Instead of using the usual longitudinal identity, we employ a generalized version where $\overline{P}_a^{\mu}$ is redefined as

$$\overline{P}_{34}^{\mu} = -\left(\frac{P_{34}^{\mu}}{P_{34}^2}\right) \to \overline{P}_{34}^{\mu} = -\left(\frac{P_{34}^{\mu}}{P_1^2 - P_2^2 + P_{34}^2}\right),\ \overline{P}_{41}^{\mu} = -\left(\frac{P_{41}^{\mu}}{P_{41}^2}\right) \to \overline{P}_{41}^{\mu} = -\left(\frac{P_{41}^{\mu}}{P_{41}^2 - P_2^2 + P_3^2}\right).$$

It is straightforward to check that under this redefinition, the factored expression in eq. (9.2) reproduces the same result as in eq. (9.1). The generalization to a higher number of points is straightforward, so, when the particles $\{P_1, P_2, P_3\}$ are off-shell, the longitudinal gluing relations that must be used in eq. (8.4) are given by

$$\sum_L A'_{2m+1}(\boldsymbol{P_r^L}, \ldots, \boldsymbol{P_2}, \ldots, \boldsymbol{P_3}, \ldots) \times A_{2q+1}^{(P_1P_k)}(P_1, \ldots, P_k^L, \ldots,) \to \sum_L \epsilon_r^{\mu\, L}\epsilon_k^{\nu\, L} = \overline{P}_r^{\mu} P_k^{\nu},$$
$$\sum_L A_{2j}^{(P_1P_k)}(P_1, \ldots, P_k^L, \ldots,) \times A'_{2i}(\boldsymbol{P_r^L}, \ldots, \boldsymbol{P_2}, \ldots, \boldsymbol{P_3}, \ldots) \to \sum_L \epsilon_k^{\mu\, L}\epsilon_r^{\nu\, L} = \overline{P}_k^{\mu} P_r^{\nu},$$

where, $P_r^{\mu} = -P_k^{\mu}$, and

$$\overline{P}_r^{\mu} = -\left(\frac{P_r^{\mu}}{P_r^2 - P_2^2 + P_3^2}\right), \qquad \overline{P}_k^{\mu} = -\left(\frac{P_k^{\mu}}{P_1^2 + P_k^2}\right). \tag{9.3}$$

[6] It is important to remind ourselves that the longitudinal contributions give the right answer only when, after applying the integration rules, all factorization channels are mediated by an off-shell vector field. This was exemplified in section 7.3.

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Thus, by applying the identities in eq. (8.5), we obtain the following simple and compact expression

$$
\begin{aligned}
&A'_{2n}(P_1,P_2,P_3,4,\ldots,2n)=\\
&\sum_{i=3}^{n}\frac{A'_{2(n-i+2)}\big(P_1,P_2,P_{3:2i-1},2i,\ldots,2n\big)\times A'_{2(i-1)}\big(P_{2i:2},P_3,4,\ldots,2i-1\big)}{s_{3:2i-1}}\\
&+\sum_{i=3}^{n+1}\frac{A'_{2(n-i+2)+1}\big(P_1,P_2,P_{3:2i-2},2i-1,\ldots,2n\big)\times A'_{2(i-1)-1}\big(P_{2i-1:2},P_3,4,\ldots,2i-2\big)}{P_1^2-P_2^2+P_{3:2i-2}^2}\\
&+(-1)\,\frac{A'_3\big(P_{4:1},P_2,P_3\big)\times A'_{2n-1}\big(P_1,P_{23},4,\ldots,2n\big)}{P_{4:1}^2-P_2^2+P_3^2}.
\end{aligned}
\tag{9.4}
$$

Obviously, when $\{P_1,P_2,P_3\}$ become on-shell, we rediscover eq. (8.6).

In order to achieve a completed recursion-relationship, it is needed to get a closed formula for the odd amplitude, $A'_{2n+1}(P_1,P_2,P_3,4,\ldots,2n+1)$. Therefore, applying the integration rules over this amplitude, one obtains the following two types of combinations

$$
\begin{aligned}
&\textbf{I.}\quad \sum_M A'_{2m+1}(\boldsymbol{P_r^M},\ldots,\boldsymbol{P_2},\ldots,\boldsymbol{P_3},\ldots)\times A_{2j}^{(P_1P_k)}(P_1,\ldots,P_k^M,\ldots,),\\
&\textbf{II.}\quad \sum_M A_{2q+1}^{(P_1P_k)}(P_1,\ldots,P_k^M,\ldots,)\times A'_{2i}(\boldsymbol{P_r^M},\ldots,\boldsymbol{P_2},\ldots,\boldsymbol{P_3},\ldots).
\end{aligned}
$$

We found that, to land on the right result by using just longitudinal degrees of freedom, the combination **I** must be glued by the relation

$$
\textbf{I.}\quad \sum_L \epsilon_r^{\mu\,L}\epsilon_k^{\nu\,L}=(-1)(P_1^2-P_2^2+P_3^3)\times\overline{P}_r^{\mu}\overline{P}_k^{\nu},
\tag{9.5}
$$

where $\overline{P}_r^{\mu}$ and $\overline{P}_k^{\nu}$ are defined in eq. (9.3), while the combination **II** has to be discarded. Note that the overall factor, $(P_1^2-P_2^2+P_3^3)$, implies that when the off-shell external particles become on-shell, the amplitude A'_{2n+1} vanishes trivially, such as it is required.

To summarize, after applying the integration rules over an even or odd amplitude, such that the factorized subamplitudes are glued only by virtual vector fields, then, we can compute this process just by considering the longitudinal degrees of freedom and the rules given in the following box

$$
\begin{array}{ccc}
A'_{2m+1}(\boldsymbol{P_r^{\epsilon}},\ldots,\boldsymbol{P_2},\ldots,\boldsymbol{P_3},\ldots)\Big|_{\epsilon_r^{\mu}\to\overline{P}_r^{\mu}} & \Longleftrightarrow^{\text{Product Allowed}} & A_{2q+1}^{(P_1P_k)}(P_1,\ldots,P_k^{\epsilon},\ldots,)\Big|_{\epsilon_k^{\mu}\to P_k^{\mu}}\\
\Big\Updownarrow{\scriptstyle \text{Product Allowed}}\ \times(-1)\,(P_1^2-P_2^2+P_3^2) & & \Big\Updownarrow{\scriptstyle \text{Product Forbidden}}\\
A_{2j}^{(P_1P_k)}(P_1,\ldots,P_k^{\epsilon},\ldots,)\Big|_{\epsilon_k^{\mu}\to\overline{P}_k^{\mu}} & \Longleftrightarrow^{\text{Product Allowed}} & A'_{2i}(\boldsymbol{P_r^{\epsilon}},\ldots,\boldsymbol{P_2},\ldots,\boldsymbol{P_3},\ldots)\Big|_{\epsilon_r^{\mu}\to P_r^{\mu}}
\end{array}
$$

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where $\overline{P}_r^{\mu}$ and $\overline{P}_k^{\nu}$ are given in eq. (9.3). Notice that the horizontal rules on the box work over the even amplitudes, i.e. $A'_{2n}(P_1, P_2, P_3, 4, \ldots, 2n)$, while the vertical rules work over the odd ones, $A'_{2n+1}(P_1, P_2, P_3, 4, \ldots, 2n+1)$.

Finally, by employing the identities in eq. (8.5) and the above box, we are able to write down a compact formula for $A'_{2n+1}(P_1, P_2, P_3, 4, \ldots, 2n+1)$,

$$\begin{aligned} A'_{2n+1}(P_1,P_2,P_3,4,\ldots,2n{+}1) = (P_1^2{-}P_2^2{+}P_3^2)\times\Bigg[\sum_{i=3}^{n+1}\left(\frac{1}{P_1^2{-}P_2^2{+}P_{3:2i-1}^2}\right) \\ \times\frac{A'_{2(n-i+2)+1}(P_1,P_2,P_{3:2i-1},2i,\ldots,2n{+}1)\times A'_{2(i-1)}(P_{2i:2},P_3,4,\ldots,2i{-}1)}{s_{3:2i-1}} \\ +\left(\frac{1}{P_{4:1}^2{-}P_2^2{+}P_3^2}\right)\times\frac{A'_3(P_{4:1},P_2,P_3)\times A'_{2n}(P_1,P_{23},4,\ldots,2n{+}1)}{s_{4:1}}\Bigg]. \end{aligned} \tag{9.6}$$

Evidently, the formulas, eqs. (9.4) and (9.6), give us a novel recursion relation, which we have checked against known results for up to ten points. The big advantage with this relation is that it is purely algebraic, as any non-linear sigma model amplitude can be decomposed to off-shell three-point amplitudes (without solving any scattering equations).

10 The soft limit and a new relation for $A_n^{\mathrm{NLSM}\oplus\phi^3}$

The soft limit for the U(N) non-linear sigma model in its CHY representation was already studied by Cachazo, Cha and Mizera (CCM) in ref. [19]. One of the main results is given by the expression (at leading order)

$$A_n(1,\ldots,n) = \epsilon\sum_{a=2}^{n-2} 2\,\tilde{k}_n\cdot k_a\, A_{n-1}^{\mathrm{NLSM}\oplus\phi^3}(1,\ldots,n-1||n-1,a,1) + \mathcal{O}(\epsilon^2), \tag{10.1}$$

where $k_n^{\mu} = \epsilon\,\tilde{k}_n^{\mu}$ and $\epsilon\to 0$.

In this section we carry out, in detail, the soft limit behaviour at six-point, but using the new recursion relation proposed in section 9. Although the generalization to a higher number of points is not straightforward, it is not complicated. We will not take into account the general case in this work.

Let us consider the amplitude, $A_6(1,2,3,4,5,6) = A'_6(5,6,1,2,3,4)$, where the soft particle is, $k_6^{\mu} = \epsilon\,\tilde{k}_6^{\mu}$, with $\epsilon\to 0$. From eq. (9.4), we have

$$\begin{aligned} A'_6(5,6,1,2,3,4) &= \frac{A'_3(5,6,P_{1:4})\times A'_5(P_{56},1,2,3,4)}{P_{56}^2} - \frac{A'_3(P_{2:5},6,1)\times A'_5(5,P_{61},2,3,4)}{P_{61}^2} \\ &\quad + \frac{A'_3(P_{3:6},1,2)\times A'_5(5,6,P_{12},3,4)}{P_{12}^2} + \frac{A'_4(5,6,P_{1:3},4)\times A'_4(P_{4:6},1,2,3)}{P_{1:3}^2} \\ &= -A'_5(P_{56},1,2,3,4) + A'_5(5,P_{61},2,3,4) - A'_5(5,6,P_{12},3,4) - \frac{2\,\epsilon\,\tilde{k}_6\cdot k_4\times A'_4(P_{456},1,2,3)}{s_{45}+2\epsilon\,\tilde{k}_6\cdot P_{45}}, \end{aligned} \tag{10.2}$$

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where the three-point building-blocks in eq. (6.3) have been used. Applying the off-shell formula proposed in eq. (9.6), it is not hard to check that, at leading order, the above five-point amplitudes become

$$-A_5'(P_{56},1,2,3,4) = (2\,\epsilon\,\tilde{k}_6\cdot k_5)\left[\frac{A_4'(P_{51},2,3,4)}{s_{51}}+\frac{A_4'(5,P_{12},3,4)}{s_{12}}\right], \tag{10.3}$$

$$A_5'(5,P_{61},2,3,4) = (2\,\epsilon\,\tilde{k}_6\cdot k_1)\left[\frac{A_4'(P_{51},2,3,4)}{s_{51}}+\frac{A_4'(5,P_{12},3,4)}{s_{12}}\right], \tag{10.4}$$

$$-A_5'(5,6,P_{12},3,4) = -(2\,\epsilon\,\tilde{k}_6\cdot P_{125})\times\frac{A_4'(5,P_{12},3,4)}{s_{12}} - 2\,\epsilon\,\tilde{k}_6\cdot k_4. \tag{10.5}$$

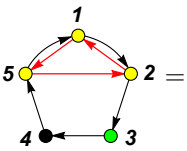
Therefore, the six-point amplitude at leading order in ϵ is given by

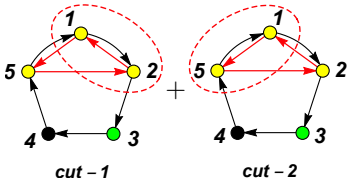
$$\begin{aligned} A_6(1,2,3,4,5,6) = {} & (2\,\epsilon\,\tilde{k}_6\cdot k_2)\left[-\frac{A_4'(P_{51},2,3,4)}{s_{15}}-\frac{A_4'(5,P_{12},3,4)}{s_{12}}\right] \\ & +(2\,\epsilon\,\tilde{k}_6\cdot k_3)\left[-\frac{A_4'(P_{51},2,3,4)}{s_{15}}\right] \\ & +(2\,\epsilon\,\tilde{k}_6\cdot k_4)\left[-\frac{A_4'(P_{51},2,3,4)}{s_{15}}-\frac{A_4'(P_{45},1,2,3)}{s_{45}}-1\right]. \end{aligned} \tag{10.6}$$

Now, from the CCM formula in eq. (10.1) one has

$$\begin{aligned} A_6(1,2,3,4,5,6) = {} & (2\,\epsilon\,\tilde{k}_6\cdot k_2)\times A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,2,1) \\ & +(2\,\epsilon\,\tilde{k}_6\cdot k_3)\times A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,3,1) \\ & +(2\,\epsilon\,\tilde{k}_6\cdot k_4)\times A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,4,1). \end{aligned} \tag{10.7}$$

Although at first glance, the eqs. (10.6) and (10.7) do not seem to be the same, notice that by choosing the gauge, $(pqr|m) = (512|3)$, the amplitude $A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,2,1)$ turns into

$A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,2,1) = \int d\mu_5^{\Lambda}$ 

$=$ 

$$\begin{aligned} & = -\frac{A_3^{\phi^3}(1,2,P_{3:5})\times A_4'(5,P_{12},3,4)}{s_{12}} - \frac{A_3^{\phi^3}(1,P_{2:4},5)\times A_4'(P_{51},2,3,4)}{s_{15}} \\ & = -\frac{A_4'(5,P_{12},3,4)}{s_{12}} - \frac{A_4'(P_{51},2,3,4)}{s_{15}}, \end{aligned} \tag{10.8}$$

where we employed the integration rules, the building-block, $A_3^{\phi^3}(P_1,P_2,P_3)=1$, and the second property from the appendix A.2. Following the same procedure, it is straightforward to see

$$A_5^{\mathrm{NLSM}\oplus\phi^3}(1,2,3,4,5||5,3,1) = -\frac{A_4'(P_{51},2,3,4)}{s_{15}}. \tag{10.9}$$

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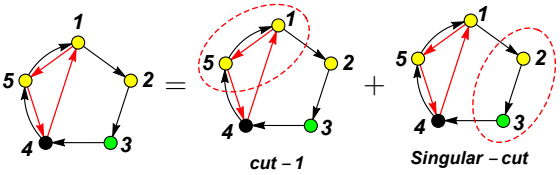
Clearly, the first two lines in eqs. (10.6) and (10.7) match perfectly, however, to compare the last lines we must take care. By direct computation, it is not hard to show that, in fact, the third lines in eqs. (10.6) and (10.7) produce the same result, but, we can extract more information from them. For example, under the gauge fixing, $(pqr|m) = (512|3)$, the amplitude $A_5^{\text{NLSM}\oplus\phi^3}(1,2,3,4,5||5,4,1)$ is given by the cuts

$$A_5^{\text{NLSM}\oplus\phi^3}(1,2,3,4,5||5,4,1) = \int d\mu_5^{\Lambda}$$



$$= -\frac{A_4'(P_{51},2,3,4)}{s_{15}} + \text{Singular-cut}. \tag{10.10}$$

Clearly, by comparing the above expression with the last line in eq. (10.6), we arrive at

$$\text{Singular-cut} = -\frac{A_4'(P_{45},1,2,3)}{s_{45}} - 1, \tag{10.11}$$

which is a simple but strong result. As it has been argued several times [13, 21] (see section 5), the integration rules, which were obtained by expanding at leading order the Λ parameter of the double cover representation, can not be applied over singular cuts. In order to achieve an extension of these rules to singular cuts, one must expand beyond leading order the Λ parameter and find a pattern, which is a highly non-trivial task. Nevertheless, eq. (10.11) tells us that the soft limit behaviour could help us to figure out this issue. This is an interesting subject to be studied in a future project.

10.1 A new relation for $A_n^{\text{NLSM}\oplus\phi^3}$

In the previous section, we observe that, using the recursion relation proposed in section 9, the soft limit behaviour of the six-point amplitude, $A_6(1,2,3,4,5,6)$, gives a factorized formula for $A_5^{\text{NLSM}\oplus\phi^3}(1,2,3,4,5||5,a,1)$ in terms of off-shell NLSM amplitudes. In this section, we are going to show a new factorization formula for the general amplitude, $A_n^{\text{NLSM}\oplus\phi^3}(1,\ldots,n||n,a,1)$.

First, let us consider the gauge fixing $(pqr|m) = (1an|2)$, so, we can suppose that the set of particles, $\{P_1, P_a, P_n\}$, are off-shell (here a is a label between $2 < a < n$). Since the $A_n^{\text{NLSM}\oplus\phi^3}(1,\ldots,n||n,a,1)$ amplitude vanishes trivially when n is even, then, it is enough to define, $n = 2m+1$. Thus, applying the integration rules with the previous setup the amplitude is factorized into

$$A_n^{\text{NLSM}\oplus\phi^3}(1,\ldots,a-1,a,a+1,\ldots,n||n,a,1) = \tag{10.12}$$
$$\sum_{i=2}^{\lfloor\frac{a}{2}\rfloor} \frac{A_{2i}'(P_{2i:n},1,2,\ldots,2i-1)\times A_{2(m-i)+3}^{\text{NLSM}\oplus\phi^3}(P_{1:2i-1},2i,\ldots,a,\ldots,n||n,a,P_{1:2i-1})}{s_{1:2i-1}} +$$
$$\sum_{i=\lceil\frac{a}{2}\rceil}^{m} \frac{A_{2i}'(\boldsymbol{P_{2i+1:1}},\boldsymbol{2},\ldots,\boldsymbol{a},\ldots,2i)\times A_{2(m-i)+3}^{\text{NLSM}\oplus\phi^3}(1,P_{2:2i},2i+1,\ldots,n||n,P_{2:2i},1)}{s_{2:2i}},$$

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where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the Floor and Ceiling functions, respectively. Notice that when $a = 3$, the first line doesn't contribute because of the properties of the Floor function.

In the particular case when $a = 2$, we choose the gauge fixing $(pqr|m) = (12n|3)$, and the factorization relation becomes

$$A_n^{\text{NLSM}\oplus\phi^3}(1,2,\ldots,n||n,2,1) = \tag{10.13}$$
$$\frac{A'_{2m}(n,P_{12},3,\ldots,n-1)\times A_3^{\text{NLSM}\oplus\phi^3}(P_{3:n},1,2||P_{3:n},2,1)}{s_{3:n}} +$$
$$\sum_{i=2}^{m}\frac{A'_{2i}(P_{2i+1:1},2,3,\ldots,2i)\times A_{2(m-i)+3}^{\text{NLSM}\oplus\phi^3}(1,P_{2:2i},2i+1,\ldots,n||n,P_{2:2i},1)}{s_{2:2i}}.$$

Clearly, when $n = 2m+1 = 5$, the relations obtained above are in agreement with the ones in eqs. (10.8) and (10.9).

11 Special Galileon theory

In ref. [18], Cachazo, He and Yuan proposed the CHY prescription to compute the S-Matrix of a special Galileon theory (sGal). The Galileon theories arise as effective field theories in the decoupling limit of massive gravity [24–26]. The special Galileon theory was discovered in refs. [18, 27] as a special class of theory with soft limits that vanish particularly fast.

As discussed previously (for more details, see ref. [18]), the CHY prescription of the sGal is given by the integral

$$A_n^{\text{sGal}} = \int \mathrm{d}\mu_n^{\text{CHY}}\,(z_{pq}z_{qr}z_{rp})^2 \times \left[\det'\mathsf{A}_n \times \det'\mathsf{A}_n\right]. \tag{11.1}$$

From this expression, it is straightforward to see the sGal is the square of the NLSM, where the product is by means of the field theory Kawai-Lewellen-Tye (KLT) kernel [28]. Schematically, one has

$$A_n^{\text{sGal}} = A_n \overset{\text{KLT}}{\otimes} A_n, \tag{11.2}$$

where the KLT matrix, usually denoted as $S[\alpha|\beta]$, is the inverse matrix of the double-color partial amplitude for the bi-adjoint ϕ^3 scalar theory [1, 3]. Notice that, from this double copy formula, we can use the whole technology developed for NLSM and apply it in sGal. Nevertheless, since our main aim is to show how the integration rules work, we will not use eq. (11.2).

11.1 A simple example

In this section, we will show how the integration rules work in a theory without partial ordering. As a simple example, we will calculate the four-point amplitude for sGal.

From eq. (3.8), the sGal in the double cover representation is given by the integral

$$A_n^{\text{sGal}} = \int \mathrm{d}\mu_n^{\Lambda}\frac{(-1)\Delta(pqr)\Delta(pqr|m)}{S_m^{\tau}} \times \left[\mathbf{det}'\mathsf{A}_n^{\Lambda} \times \mathbf{det}'\mathsf{A}_n^{\Lambda}\right], \tag{11.3}$$

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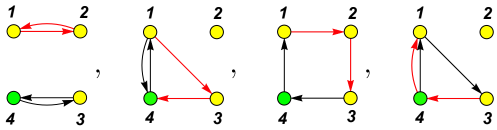
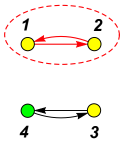
where we have defined, $\mathbf{det}'\mathsf{A}_n^\Lambda = \prod_{a=1}^n \frac{(y\sigma)_a}{y_a} \times \det'\mathsf{A}_n^\Lambda$. After choosing a gauge fixing, by the *rule-I* in section 5 we know that the Faddeev-Popov factor goes as, $\frac{(-1)\Delta(pqr)\Delta(pqr|m)}{S_m^\tau} \sim \Lambda^{-4} + \mathcal{O}(\Lambda^{-2})$, (eq. (5.4)). Thus, in order to cancel this Λ^{-4} factor, at leading order, a cut-contribution in the special Galileon theory must cut at least one arrow of each reduced determinant, this fact comes from table 1. This is summarized in *Rule-II*. For example, for the four-point amplitude, $A_4^{\rm sGal}(1,2,3,4)$, let us consider the following four different setups

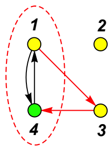


(11.4)

where the red/black arrows denote a given reduced determinant. Clearly, the first two graphs with reduced matrices, $(\mathsf{A}_4^\Lambda)_{12}^{12} \times (\mathsf{A}_4^\Lambda)_{34}^{34}$ and $(\mathsf{A}_4^\Lambda)_{34}^{13} \times (\mathsf{A}_4^\Lambda)_{14}^{14}$, respectively, have the following singular cuts



$$\rightarrow \mathbf{det}'\mathsf{A}_4^\Lambda \times \mathbf{det}'\mathsf{A}_4^\Lambda\Big|_{34}^{12} \sim \Lambda^0\,,$$



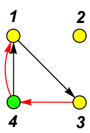
$$\rightarrow \mathbf{det}'\mathsf{A}_4^\Lambda \times \mathbf{det}'\mathsf{A}_4^\Lambda\Big|_{23}^{41} \sim \Lambda^2\,.$$

On the other hand, the third and fourth graphs do not have any singular cuts, therefore, we can apply the integration rules over them.

11.1.1 The four-point computation

To carry out the four-point sGal amplitude, we choose the fourth setup in eq. (11.4). Thus, from the integration rules, we have three cut contributions given by

$$A_4^{\rm sGal}(1,2,3,4) = \int d\mu_4^\Lambda$$



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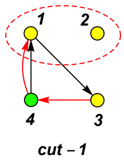

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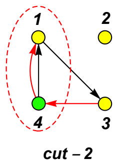

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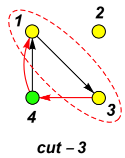
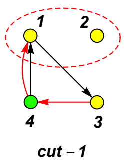


. (11.5)

It is straightforward to see that the first contribution vanishes trivially,



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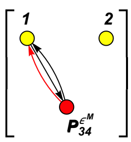
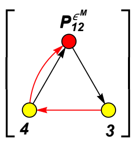
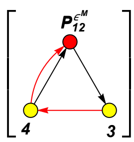


$$\times \left(\frac{1}{s_{34}}\right) \times$$



$$= \sum_M (\sigma_{12}\sigma_{2P_{34}}\sigma_{P_{34}1})^2 \times$$

$$\mathrm{PT}(1,P_{34}) \det\left[(\mathsf{A}_3)_{1P_{34}}^{1P_{34}}\right] \times \frac{1}{\sigma_{P_{34}1}} \det\left[(\mathsf{A}_3)_1^{P_{34}}\right]\Big|_{P_{34}\rightarrow\frac{\epsilon_{34}^M}{\sqrt{2}}} \times \left(\frac{1}{s_{34}}\right) \times$$



$= 0,$

where we used the identity, $\det\left[(\mathsf{A}_3)_{1P_{34}}^{1P_{34}}\right] = 0$. The first and second reduced determinants correspond to the black and red arrows, respectively. In the following, we associate the first reduced determinant with the black arrows, and the second reduced determinant with

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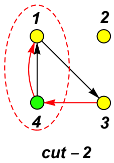
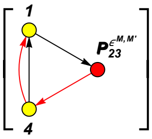
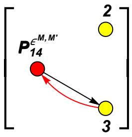
the red arrows. By a similar computation, the *cut-3* also vanishes, then, the only non-zero contribution comes from the *cut-2*.



$=$



$\times \left(\frac{1}{s_{14}}\right) \times$



$$= \sum_{M,M'} (\sigma_{41}\sigma_{1P_{23}}\sigma_{P_{23}4})^2 \times$$

$$\left[\frac{1}{\sigma_{41}\sigma_{1P_{23}}} \det\left[(\mathsf{A}_3)^{41}_{1P_{23}}\right]\Big|_{P_{23}\to\frac{\epsilon^{M}_{23}}{\sqrt{2}}} \times \frac{1}{\sigma_{P_{23}4}\sigma_{41}} \det\left[(\mathsf{A}_3)^{P_{23}4}_{41}\right]\Big|_{P_{23}\to\frac{\epsilon^{M'}_{23}}{\sqrt{2}}}\right] \times \left(\frac{1}{s_{14}}\right) \times$$

$$(\sigma_{23}\sigma_{3P_{14}}\sigma_{P_{14}2})^2 \times \left[\frac{1}{\sigma_{P_{14}3}} \det\left[(\mathsf{A}_3)^{P_{14}}_{3}\right]\Big|_{P_{14}\to\frac{\epsilon^{M}_{14}}{\sqrt{2}}} \times \frac{1}{\sigma_{3P_{14}}} \det\left[(\mathsf{A}_3)^{3}_{P_{14}}\right]\Big|_{P_{14}\to\frac{\epsilon^{M'}_{14}}{\sqrt{2}}}\right]$$

$$= -s_{12}\, s_{13}\, s_{14} \ ,$$

where the completeness identities, $\sum_M \epsilon^{\mu\, M}_{23}\epsilon^{\nu\, M}_{14} = \eta^{\mu\nu}$ and $\sum_{M'} \epsilon^{\mu\, M'}_{23}\epsilon^{\nu\, M'}_{14} = \eta^{\mu\nu}$, have been used. Therefore, we obtain

$$A^{\rm sGal}_4(1,2,3,4) = -s_{12}\, s_{13}\, s_{14}, \tag{11.6}$$

which is the right answer.

Finally, it is straightforward to generalize this simple example to a higher number of points. Additionally, it would be interesting to understand the properties of the special Galileon theory similar to ones obtained for NLSM in sections 7.3, 8.1 and 9.

12 Conclusions

The double-cover version of the CHY formalism is an intriguing extension that sheds new light on how scattering amplitudes can emerge as factorized pieces. Focusing on the non-linear sigma model, we have illustrated how unphysical channels appear at intermediate steps, always canceling in the end, and thus producing the right answer. The origin of factorizations is the appearance of one “free” scattering equation. This is the origin of the off-shell channel through which the amplitudes factorize.

We have analyzed the factorizations obtained in the non-linear sigma model because they perfectly illustrate the mechanism, and the cancellations that eventually render the full result free of unphysical poles. For this theory, we have obtained three different factorization relationships, two of them emerged naturally from the double-cover framework (by using the A_{2n} and A'_{2n} prescriptions), while the other one was obtained fortuitously by considering the longitudinal degrees of freedom of the cut-contributions from the new A'_{2n} prescription. By comparing to BCFW on-shell recursion relations we have found a perfect correspondence between the unphysical terms of the double-cover formalism and terms that arise from poles at infinity in the BCFW formalism. In that sense, the double-cover version of CHY succeeds in evaluating what appears as poles at infinity in BCFW recursion as simple CHY-type integrals of the double cover. It would be interesting if this correspondence could be made more explicit. Certainly, it hints at the possibility that

an alternative formulation of the problem of poles at infinity in BCFW recursion exists, without recourse to the particular double-cover formalism.

Using the new prescription for the reduced determinant in the integrand, we found a factorization relation where all the intermediate off-shell particles are spin-1 (gluons). The corresponding momenta in the reduced determinants are replaced by polarization vectors. We would like to investigate further the connection between this new object and the integrand for generalized Yang-Mills-Scalar theory [18]. At first sight, we thought that this new matrix could be related to the novel model proposed by Cheung, Remmen, Shen, and Wen in [29, 30], nevertheless, after comparing the numerators at the four-point computation, the relation among these two approaches is unclear.

On the other hand, when we replaced the off-shell gluons with only the longitudinal degrees of freedom, we were able to rewrite the factorized pieces in terms of lower-point NLSM amplitudes in the new prescription, with up to three off-shell punctures. This is a very surprising result, and understanding the origin of this connection is left for future work. The big advantage of being able to rewrite the factorized pieces is that we can iteratively promote the lower-point NLSM amplitudes to the double cover, which would lead to further factorization. Thus, any NLSM amplitude can be factorized entirely in terms of off-shell three-point amplitudes. This is a novel off-shell recursion relation. The resulting expression is algebraic, and no scattering equation needs to be solved. We have checked the validity of the recursion relation up to ten points (17 points for odd amplitudes). We would like to find the connection between the recursion relation and Berends-Giele currents [20, 31–37].

The novel recursion relation can also be used to investigate singular cuts and NLSM$\oplus\phi^3$ amplitudes through the soft limit. CCM showed how the soft limit of an NLSM amplitude can be expressed in terms of NLSM $\oplus$ ϕ^3 amplitudes [19]. We calculated the soft limit of a six-point NLSM amplitude in two ways, using the CCM formula and using the novel recursion relation. This gives a relation for a specific singular cut. Further investigations into the nature of the soft limits might reveal insight into the singular cuts in general. Also, we were able to find a factorization relation for the NLSM $\oplus$ ϕ^3 amplitudes.

Lastly, we showed how the special Galileon amplitudes can be calculated in a double cover language. One intriguing feature is that for some configurations, the off-shell particle propagating between the lower-point pieces is spin-2 (graviton). So, we have observed that for the NLSM, off-shell gluons appear, while for the special Galileon theory, both off-shell gluons and gravitons appear. This might be connected to the fact that the NLSM originated as an effective theory of pion scattering, while the Galileon theories arise as effective field theories in the decoupling limit of massive gravity. This also seems natural, as the special Galileon theory is the square of the NLSM, using the KLT relation.

It seems evident that there are numerous aspects of CHY on a double cover that need to be investigated.

Acknowledgments

We would like to thank N.E.J. Bjerrum-Bohr and P.H. Damgaard for valuable discussions. We are grateful to C. Vergu for pointing out the Desnanot-Jacobi identity. This work

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was supported in part by the Danish National Research Foundation (DNRF91). The work of H.G. is supported by the Niels Bohr Institute — University of Copenhagen and the Santiago de Cali University (USC).

A Some matrix identities

In this section, we are going to provide some useful properties of the determinant of the A_n matrix. Although we lack formal proofs for many of the relations, we have performed numerous checks, up to ten points.

A.1 A new NLSM prescription from CHY

In this appendix, we formulate two propositions which have been employed to redefine the n-point NLSM amplitude from the CHY framework.

Proposition 1. Let M be a $2n \times 2n$ antisymmetric matrix. Then M satisfy the identity

$$\mathrm{Pf}\Big[(M)^{ik}_{ik}\Big] \times \mathrm{Pf}\Big[(M)^{kj}_{kj}\Big] = \det\Big[(M)^{ik}_{kj}\Big], \tag{A.1}$$

up to an overall sign.

Proof. We start with the Desnanot-Jacobi identity [38], given by

$$\det[M]\,\det\Big[(M)^{ij}_{ij}\Big] = \det\big[(M)^{i}_{i}\big]\,\det\Big[(M)^{j}_{j}\Big] - \det\big[(M)^{i}_{j}\big]\,\det\Big[(M)^{j}_{j}\Big]. \tag{A.2}$$

Now, let M be a $2n \times 2n$ antisymmetric matrix, therefore, $(M)^{k}_{k}$ is a $(2n-1) \times (2n-1)$ antisymmetric matrix. Thus, from the identity in eq. (A.2), it is straightforward to see that

$$0 = \det\Big[(M)^{ki}_{ki}\Big]\,\det\Big[(M)^{kj}_{kj}\Big] - \det\Big[(M)^{ki}_{kj}\Big]\,\det\Big[(M)^{kj}_{ki}\Big], \tag{A.3}$$

where we used the fact, $\det\big[(M)^{k}_{k}\big] = \det\Big[(M)^{kij}_{kij}\Big] = 0$. Since, $[(M)^{kj}_{ki}] = [(M)^{ki}_{kj}]^{\mathrm{t}} = -[(M)^{ki}_{kj}]$, then

$$\Big\{\mathrm{Pf}\Big[(M)^{ik}_{ik}\Big] \times \mathrm{Pf}\Big[(M)^{kj}_{kj}\Big]\Big\}^2 = \Big\{\det\Big[(M)^{ik}_{kj}\Big]\Big\}^2, \tag{A.4}$$

and proposition 1 has been proved.

Proposition 2. Let A be the antisymmetric matrix defined in eq. (2.6). When its size is $(2n+1) \times (2n+1)$, then

$$\det\Big[(\mathsf{A})^{ik}_{kj}\Big] = 0. \tag{A.5}$$

Proof. Let us consider the $2n \times 2n$ antisymmetric matrix given by $(\mathsf{A})^{k}_{k}$. Thus, from the Desnanot-Jacobi identity in eq. (A.2), one has

$$\det\Big[(\mathsf{A})^{k}_{k}\Big] \times \det\Big[(\mathsf{A})^{kij}_{kij}\Big] = -\Big\{\det\Big[(\mathsf{A})^{ik}_{kj}\Big]\Big\}^2, \tag{A.6}$$

where we used, $\det\Big[(\mathsf{A})^{ki}_{ki}\Big] = \det\Big[(\mathsf{A})^{kj}_{kj}\Big] = 0$. Under the support of the scattering equations, $S_a = 0$, and the on-shell conditions, $k_a^2 = 0$, it is simple to show that the A matrix has co-rank 2, therefore, $\det\Big[(\mathsf{A})^{k}_{k}\Big] = 0$. This implies that, $\det\Big[(\mathsf{A})^{ik}_{kj}\Big] = 0$, and the proof is completed.

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A.2 Off-shell determinant properties

In this appendix we give some properties of the determinant when there is an off-shell particle. These properties involve the matrices, A_n and $\mathsf{A}_n|_{P_i\to\frac{1}{\sqrt{2}}\epsilon_i}$.

This is very important to remark that those properties are supported on the solution of the scattering equations, and, although we do not have a formal proof, they have been checked up to ten points.

Let us consider n-particles with momenta, $(P_1, P_2, P_3, k_4, \ldots, k_n)$, where the first three are off-shell, i.e. $P_i^2 \neq 0$, and the momentum conservation condition is satisfied, $P_1 + P_2 + P_3 + k_4 \cdots + k_n = 0$. Additionally, the three off-shell punctures are fixed, $\sigma_{P_1} = \mathrm{c}_1$, $\sigma_{P_2} = \mathrm{c}_2$, $\sigma_{P_3} = \mathrm{c}_3$, $\mathrm{c}_i \in \mathbb{C}$, where $\mathrm{c}_1 \neq \mathrm{c}_2 \neq \mathrm{c}_3$. Thus, the "$n-3$" scattering equations are given by

$$S_a = \frac{2\,k_a\cdot P_1}{\sigma_{aP_1}} + \frac{2\,k_a\cdot P_2}{\sigma_{aP_2}} + \frac{2\,k_a\cdot P_3}{\sigma_{aP_3}} + \sum_{\substack{b=4\\a\neq b}}^{n}\frac{2\,k_a\cdot k_b}{\sigma_{ab}} = 0, \quad a = 4,\ldots,n. \tag{A.7}$$

Properties. Under the support of the scattering equations and using the above setup, we have the following properties

I. Let n an odd number, $n = 2m+1$, then

$$\det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right] = (P_1^2 - P_2^2 - P_3^2)\times\frac{(-1)}{\sigma_{P_2\,P_3}}\det\left[(\mathsf{A}_n)^{P_1P_2}_{P_2P_3}\right]. \tag{A.8}$$

Notice that if all particles are on-shell, $P_i^2 = 0$, the right hand side vanishes trivially by the overall factor, $(P_1^2 - P_2^2 - P_3^2)$.

When the momentum P_1^μ is replaced by an off-shell polarization vector, $P_1^\mu \to \frac{1}{\sqrt{2}}\epsilon_1^\mu$, $(\epsilon_1\cdot P_1 \neq 0)$, the identity keeps the same form, namely

$$\det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right]\Big|_{P_1^\mu\to\frac{1}{\sqrt{2}}\epsilon_1^\mu} = (P_1^2 - P_2^2 - P_3^2)\times\frac{(-1)}{\sigma_{P_2P_3}}\det\left[(\mathsf{A}_n)^{P_1P_2}_{P_2P_3}\right]\Big|_{P_1^\mu\to\frac{1}{\sqrt{2}}\epsilon_1^\mu}. \tag{A.9}$$

This identity is no longer satisfied if there are two off-shell polarization vectors.

II. Let n an even number, $n = 2m$, then

$$\begin{aligned}\frac{(-1)}{\sigma_{P_1P_2}}\det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right] &= -(P_1^2 + P_2^2 + P_3^2)\times\frac{1}{\sigma_{P_1P_2}\,\sigma_{P_2P_3}}\det\left[(\mathsf{A}_n)^{P_1P_2}_{P_2P_3}\right]\\ &= -(P_1^2 + P_2^2 + P_3^2)\times\frac{(-1)}{\sigma_{P_1P_2}\,\sigma_{P_2P_1}}\det\left[(\mathsf{A}_n)^{P_1P_2}_{P_2P_1}\right].\end{aligned} \tag{A.10}$$

If all particles are on-shell, $P_i^2 = 0$, the right hand side vanishes trivially by the overall factor, $(P_1^2 + P_2^2 + P_3^2)$.

When the momentum P_1^μ is replaced by an off-shell polarization vector, $P_1^\mu \to \frac{1}{\sqrt{2}}\epsilon_1^\mu$, $(\epsilon_1\cdot P_1 \neq 0)$, then, eq. (A.10) is no longer an identity. Instead, we have a new identity given by

$$\frac{(-1)}{\sigma_{P_1\,P_2}}\det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right]\Big|_{P_1^\mu\to\frac{1}{\sqrt{2}}\epsilon_1^\mu} = \frac{1}{\sigma_{P_1\,P_3}}\det\left[(\mathsf{A}_n)^{P_1}_{P_3}\right]\Big|_{P_1^\mu\to\frac{1}{\sqrt{2}}\epsilon_1^\mu}. \tag{A.11}$$

If there are two off-shell polarization vectors, then, this equality is no longer true.

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III. Let n an odd number, $n = 2m+1$, and let us consider the particles P_2 and P_3 on-shell ($P_2^2 = P_3^2 = 0$). Then, we have the following identities

$$\frac{1}{\sigma_{P_1\,P_3}} \det\left[(\mathsf{A}_n)^{P_1}_{P_1}\right] = \frac{(-1)}{\sigma_{P_2\,P_3}} \det\left[(\mathsf{A}_n)^{P_1}_{P_2}\right], \tag{A.12}$$

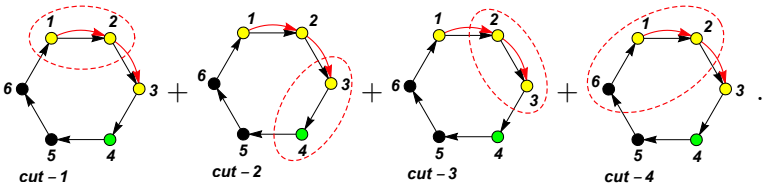
$$\det\left[(\mathsf{A}_n)^{P_1}_{P_1}\right] = \left[P_1^2 \times \frac{1}{\sigma_{P_2\,P_3}}\right]^2 \det\left[(\mathsf{A}_n)^{P_1\,P_2\,P_3}_{P_1\,P_3\,P_3}\right]. \tag{A.13}$$

B Six-point computations

In this section we are going to explicitly calculate the six-point NLSM amplitudes $A'_6(\mathbb{I}^{(123)})$, $A'_6(\mathbb{I}^{(134)})$ and $A_6(\mathbb{I}^{(13)})$, where the two first are defined with the new integrand prescription, while the third is defined with the standard integrand. We will calculate some of the cut-contributions in detail, with the hope that the reader becomes more familiar with the double cover formalism. The rest of the cut-contributions can be computed in a similar way.

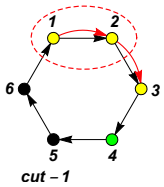
B.1 $A'_6(\mathbb{I}^{(123)})$

Let us consider the six-point NLSM amplitude, $\mathcal{A}_6(1,2,3,4,5,6)$, with the gauge fixing, $(pqr|m) = (123|4)$, and the reduced matrix $[\mathsf{A}_n]^{12}_{23}$ (i.e. $(ijk) = (123)$). Applying *rule-I*, this amplitude has the following contributions

$A'_6(\mathbb{I}^{(123)}) =$  . (B.1)

We will compute in detail the first contribution, which we call *cut-1*. The other cuts can be evaluated using the same techniques.

From the *integration rules*, *cut-1* is evaluated as

 $= \sum_M \dfrac{A'_3(1,2,P^M_{3:6}) \times A_5^{(P_{12}\,3)}(P^M_{12},3,4,5,6)}{s_{3:6}}$. (B.2)

The three-point amplitude was already computed in eq. (6.2). We remind ourselves that the notation $P^M_{3:6}$ means that the off-shell momentum, $P^\mu_{3:6}$, must be replacement by the polarization vector, $P^\mu_{3:6} \to \frac{1}{\sqrt{2}}\epsilon^{M\,\mu}_{3:6}$. More precisely, the three-point amplitude becomes

$$A'_3(1,2,P^M_{3:6}) = \sqrt{2}\,(\epsilon^M_{3:6}\cdot k_1)\,. \tag{B.3}$$

Before computing the five-point amplitude in eq. (B.2), it is useful to use the identity, $A_5^{(P_{12}\,3)}(P^M_{12},3,4,5,6) = P^2_{12} \times A'_5(P^M_{12},3,4,5,6)$. Thus, by applying the integration rules

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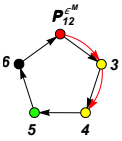
for $A'_5(P_{12}^M,3,4,5,6)$ one has

$$A'_5(P_{12}^M,3,4,5,6)=\int d\mu_5^{\Lambda}\;\raisebox{-0.5ex}{  }\;=\sum_N\Bigg\{\frac{A'_3(P_{12}^M,3,P_{4:6}^N)\times A_4^{(P_{1:3}\,4)}(P_{1:3}^N,4,5,6)}{s_{4:6}}+$$
$$\frac{A_4^{(P_{12}P_{34})}(P_{12}^M,P_{34}^N,5,6)\times A'_3(P_{5:2}^N,3,4)}{s_{56P_{12}}}+\frac{A'_4(P_{12}^M,3,P_{45}^N,6)\times A_3^{(P_{6:3}\,4)}(P_{6:3}^N,4,5)}{s_{45}}\Bigg\},\tag{B.4}$$

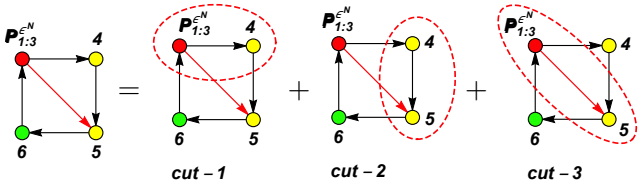
with $\sum_N \epsilon_i^{N\,\mu}\epsilon_j^{N\,\nu}=\eta^{\mu\nu}$. From the building blocks in eqs. (6.2) and (6.3), the above three-point amplitudes are straightforward to compute. We find that

$$A'_3(P_{12}^M,3,P_{4:6}^N)=\epsilon_{12}^M\cdot\epsilon_{4:6}^N,\quad A'_3(P_{5:2}^N,3,4)=\sqrt{2}\,\epsilon_{5:2}^N\cdot k_4,\quad A_3^{(P_{6:3}\,4)}(P_{6:3}^N,4,5)=\sqrt{2}\,s_{45}\,(\epsilon_{6:3}^N\cdot k_5).\tag{B.5}$$

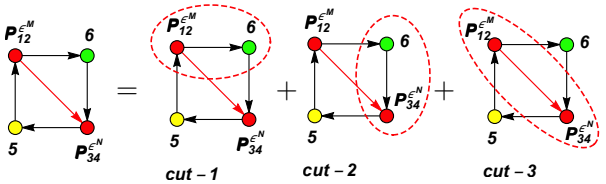
Next, using the same procedure as in eq. (7.8), we evaluate the four-point graph, $A'_4(P_{12}^M,3,P_{45}^N,6)$, arriving at

$$A'_4(P_{12}^M,3,P_{45}^N,6)=2(\epsilon_{12}^M\cdot k_6)\,(\epsilon_{45}^N\cdot k_6)\left(\frac{1}{s_{6P_{45}}}+\frac{1}{s_{6P_{12}}}\right).\tag{B.6}$$

On the other hand, in order to avoid singular cuts when applying the integration rules over $A_4^{(P_{1:3}\,4)}(P_{1:3}^N,4,5,6)$, we employ the identity, $A_4^{(P_{1:3}\,4)}(P_{1:3}^N,4,5,6)=A_4^{(P_{1:3}\,5)}(P_{1:3}^N,4,5,6)$. Thus,

$$A_4^{(P_{1:3}\,5)}(P_{1:3}^N,4,5,6)=\int d\mu_4^{\Lambda}\;\raisebox{-0.5ex}{  }\;\begin{matrix}\\ \text{cut}-1\end{matrix}\quad\begin{matrix}\\ \text{cut}-2\end{matrix}\quad\begin{matrix}\\ \text{cut}-3\end{matrix}$$
$$=-\sqrt{2}\,s_{46}(\epsilon_{1:3}^N\cdot k_4)-\frac{\sqrt{2}\,s_{46}\,s_{45}\,(\epsilon_{1:3}^N\cdot k_6)}{s_{6P_{1:3}}}-\sqrt{2}\,s_{46}\,(\epsilon_{1:3}^N\cdot k_5),\tag{B.7}$$

where we again have used the three-point building blocks in eqs. (6.2) and (6.3). Lastly, since for the amplitude, $A_4^{(P_{12}\,P_{34})}(P_{12}^M,P_{34}^N,5,6)$, the above identity is no longer valid, namely[7] $A_4^{(P_{12}\,P_{34})}(P_{12}^M,P_{34}^N,5,6)\neq A_4^{(P_{12}\,5)}(P_{12}^M,P_{34}^N,5,6)$, we make use of the BCJ relation [6, 9], $s_{65}\,\mathrm{PT}(5,6,P_{12},P_{34})+s_{6P_{125}}\,\mathrm{PT}(5,P_{12},6,P_{34})=0$. From this we obtain the equality $A_4^{(P_{12}\,P_{34})}(P_{12}^M,P_{34}^N,5,6)=\left(\frac{s_{6P_{34}}}{s_{56}}\right)\times A_4^{(P_{12}\,P_{34})}(P_{12}^M,6,P_{34}^N,5)$. Now, applying the integration rules, one has

$$A_4^{(P_{12}\,P_{34})}(P_{12}^M,P_{34}^N,5,6)=\left(\frac{s_{6P_{34}}}{s_{56}}\right)\times\int d\mu_4^{\Lambda}\;\raisebox{-0.5ex}{  }\;\begin{matrix}\\ \text{cut}-1\end{matrix}\quad\begin{matrix}\\ \text{cut}-2\end{matrix}\quad\begin{matrix}\\ \text{cut}-3\end{matrix}$$
$$=-\frac{2\,s_{6P_{34}}\,(\epsilon_{12}^M\cdot k_6)\,(\epsilon_{34}^N\cdot k_5)}{s_{6P_{12}}}-2\,(\epsilon_{12}^M\cdot k_5)\,(\epsilon_{34}^N\cdot k_6)-s_{6P_{34}}(\epsilon_{12}^M\cdot\epsilon_{34}^N).\tag{B.8}$$

[7] This is because there is more than one off-shell polarization vector.

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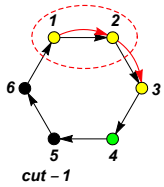
Utilizing the results obtained in eqs. (B.5) to (B.8) , it is straightforward to check the five-point amplitude, $A_5^{(P_{12}\,3)}(P_{12}^M,3,4,5,6)$, is given by

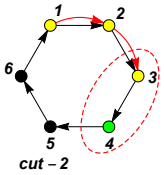
$$A_5^{(P_{12}\,3)}(P_{12}^M,3,4,5,6) = -s_{12}\sqrt{2}\left\{\frac{s_{46}}{s_{4:6}}\left[(\epsilon_{12}^M\cdot k_4)+\frac{s_{45}(\epsilon_{12}^M\cdot k_6)}{s_{6P_{1:3}}}+(\epsilon_{12}^M\cdot k_5)\right]\right.$$
$$\left.+\frac{s_{6P_{34}}}{s_{56P_{12}}}\left[\frac{s_{45}\,(\epsilon_{12}^M\cdot k_6)}{s_{6P_{12}}}+\frac{s_{46}\,(\epsilon_{12}^M\cdot k_5)}{s_{6P_{34}}}+(\epsilon_{12}^M\cdot k_4)\right]-s_{56}\,(\epsilon_{12}^M\cdot k_6)\left[\frac{1}{s_{6P_{45}}}+\frac{1}{s_{6P_{12}}}\right]\right\}, \tag{B.9}$$

and therefore *cut-1* in eq. (B.2) is given by

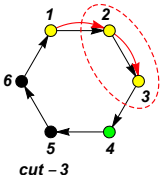


$$= \sum_M \frac{A_3'(1,2,P_{3:6}^M)\times A_5^{(P_{12}\,3)}(P_{12}^M,3,4,5,6)}{s_{3:6}} = -\left\{\frac{s_{46}}{s_{4:6}}\left[s_{14}+\frac{s_{45}s_{16}}{s_{6P_{1:3}}}+s_{15}\right]\right.$$
$$\left.+\frac{s_{6P_{34}}}{s_{56P_{12}}}\left[\frac{s_{45}\,s_{16}}{s_{6P_{12}}}+\frac{s_{46}\,s_{15}}{s_{6P_{34}}}+s_{14}\right]-s_{56}\,s_{16}\left[\frac{1}{s_{6P_{45}}}+\frac{1}{s_{6P_{12}}}\right]\right\}. \tag{B.10}$$

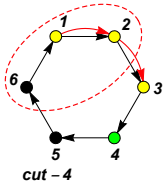
The other contributions, *cut-2,3,4*, are calculated in a similar fashion. We find that



$$= \sum_M \frac{A_5'(1,2,P_{34}^M,5,6)\times A_3^{(P_{5:2}\,3)}(P_{5:2}^M,3,4)}{s_{34}} = -\left\{\frac{s_{15}}{s_{5:1}}\left[s_{14}+\frac{s_{45}s_{16}}{s_{5P_{2:4}}}+s_{46}\right]\right.$$
$$\left.+\frac{s_{5P_{12}}}{s_{56P_{34}}}\left[\frac{s_{45}\,s_{16}}{s_{5P_{34}}}+\frac{s_{46}\,s_{15}}{s_{5P_{12}}}+s_{14}\right]-s_{56}\,s_{45}\left[\frac{1}{s_{5P_{16}}}+\frac{1}{s_{5P_{34}}}\right]\right\}, \tag{B.11}$$



$$= \sum_M \frac{A_3'(P_{4:1}^M,2,3)\times A_5^{(1\,P_{23})}(1,P_{23}^M,4,5,6)}{s_{4:1}} = -\left\{\frac{s_{46}}{s_{4:6}}\left[s_{34}+\frac{s_{45}s_{36}}{s_{6P_{1:3}}}+s_{35}\right]\right.$$
$$\left.-\frac{s_{56}\,s_{36}}{s_{6P_{45}}}+\frac{s_{6P_{2:4}}\,s_{34}}{s_{5:1}}\right\}, \tag{B.12}$$



$$= \sum_M \frac{A_4'(1,2,P_{3:5}^M,6)\times A_4^{(P_{6:2}\,3)}(P_{6:2}^M,3,4,5)}{s_{3:5}} = -\frac{s_{16}\,s_{35}}{s_{3:5}}\times$$
$$\left(\frac{1}{s_{16}}+\frac{1}{s_{6P_{3:5}}}\right)\times\left(s_{36}+\frac{s_{34}\,s_{56}}{s_{5P_{6:2}}}+s_{46}\right). \tag{B.13}$$

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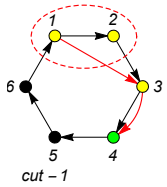
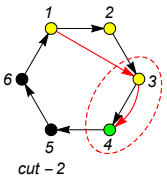
B.2 $A_6'(\mathbb{I}^{(134)})$

In this section, we just write down the results found for the cut-contributions obtained in eq. (7.26). Using the same method presented above, it is straightforward to arrive



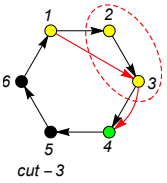
$$= \sum_M \frac{A_3^{(1\,P_{3:6})}(1,2,P_{3:6}^M)\times A_5'(P_{12}^M,3,4,5,6)}{s_{3:6}} = \frac{s_{46}}{s_{4:6}}\left[s_{24}+\frac{s_{45}s_{26}}{s_{6P_{1:3}}}+s_{25}\right]$$

$$+\frac{s_{6P_{34}}}{s_{56P_{12}}}\left[\frac{s_{45}\,s_{26}}{s_{6P_{12}}}+\frac{s_{46}\,s_{25}}{s_{6P_{34}}}+s_{24}\right]-s_{56}\,s_{26}\left[\frac{1}{s_{6P_{45}}}+\frac{1}{s_{6P_{12}}}\right], \tag{B.14}$$

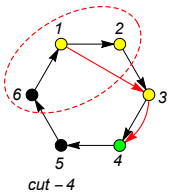


$$= \sum_M \frac{A_5^{(1\,P_{34})}(1,2,P_{34}^M,5,6)\times A_3'(P_{5:2}^M,3,4)}{s_{34}} = -\frac{s_{26}\,s_{56}\,s_{45}}{s_{5P_{34}}\,s_{6P_{3:5}}}+\frac{s_{24}\,s_{6P_{2:4}}}{s_{5:1}}$$

$$+\frac{1}{s_{P_{34}56}}\left[s_{25}\,s_{46}+\frac{s_{26}\,s_{6P_{34}}\,s_{45}}{s_{6P_{12}}}+s_{24}\,s_{6P_{34}}\right], \tag{B.15}$$



$$= \frac{A_3(3,P_{4:1},2)\times A_5'(1,P_{23},4,5,6)}{s_{4:1}} = 0, \tag{B.16}$$



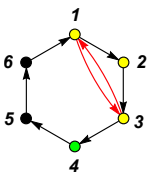
$$= \sum_M \frac{A_4^{(1\,P_{3:5})}(1,2,P_{3:5}^M,6)\times A_4'(P_{6:2}^M,3,4,5)}{s_{3:5}} = -\frac{s_{26}\,s_{45}}{s_{3:5}}\times$$

$$\left(\frac{1}{s_{45}}+\frac{1}{s_{5P_{6:2}}}\right)\times\left(s_{15}+\frac{s_{12}\,s_{56}}{s_{6P_{3:5}}}+s_{25}\right). \tag{B.17}$$

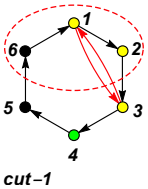
B.3 $A_6(\mathbb{I}^{(13)})$

Now, we focus on applying the *integration rules* for $A_6(\mathbb{I}^{(13)})$. We recall that this notation means that the reduced Pfaffian is given by $-\mathrm{PT}^T(1,3)\times\det[(\mathsf{A}_6^\Lambda)_{13}^{13}]$. In addition, such as in the previous examples, we fix the gauge by $(pqr|m)=(123|4)$. Thus, from the eq. (7.19), we have that

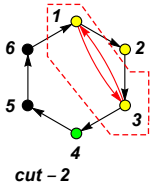
$$A_6(\mathbb{I}^{(13)}) = \int d\mu_6^\Lambda$$



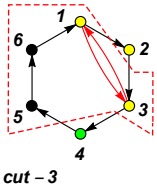
$$=$$



$$+$$



$$+$$



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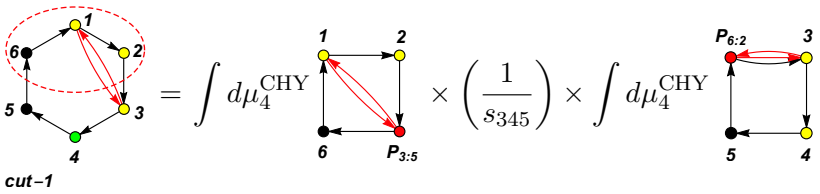
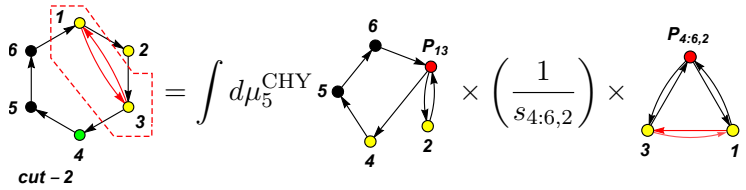
Applying the *integration rules*, *cut-1* is split into



$$= \frac{A_4(\mathbf{1},2,\boldsymbol{P_{3:5}},6)\times A_4(\boldsymbol{P_{6:2}},\mathbf{3},4,5)}{s_{3:5}} = \frac{s_{26}\,s_{35}}{s_{3:5}}\,. \tag{B.18}$$

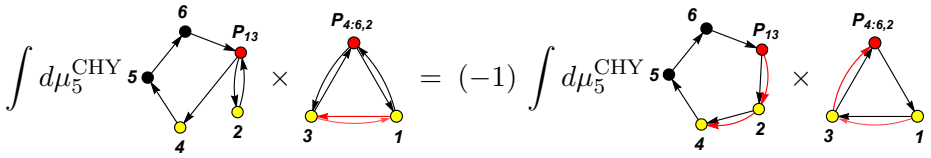
On the last equality we used the identity, $A_4(\boldsymbol{P_{6:2}},\mathbf{3},4,5) = A_4(\boldsymbol{P_{6:2}},3,\mathbf{4},5)$ (in order to avoid singular cuts), and the same procedure as in eq. (7.2). This identity is supported over the off-shell Pfaffian properties given in appendix A.2.

The following contribution is the *cut-2* (*strange-cut*), which, by the *integration rules*, is broken as



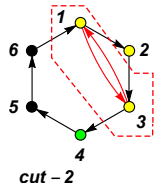
(B.19)

Notice that on the first graph the our method can not be employed. Nevertheless, similar to Yang-Mills theory [21], this strange-cut can be rewritten in the following way



$$= (-1)\, A_5'(P_{13},2,4,5,6)\times A_3'(1,3,P_{4:6,2}), \tag{B.20}$$

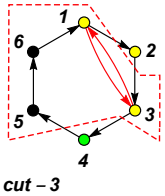
where we used the identities formulated in appendix A.2. Therefore, this cut turns into



$$= (-1)\,\frac{A_5'(P_{13},2,4,5,6)\times A_3'(1,3,P_{4:6,2})}{s_{4:6,2}} = s_{13}\left[\frac{s_{46}}{s_{456}} + \frac{s_{26}+s_{46}}{s_{56P_{13}}}\right]. \tag{B.21}$$

The five-point amplitude, $A_5'(P_{13},2,4,5,6)$, was already calculated in eq. (B.4) and the three-point function is given in eq. (6.2).

Lastly, the strange *cut-3* is



$$= (-1)\,\frac{A_3'(P_{5:1,3},2,4)\times A_5'(1,3,P_{24},5,6)}{s_{24}} = s_{24}\left[\frac{s_{26}+s_{46}}{s_{56P_{24}}} + \frac{s_{26}+s_{36}+s_{46}}{s_{561}}\right]. \tag{B.22}$$

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B.4 Longitudinal contributions

In this section, we consider just the longitudinal degrees of freedom of all cut-contributions obtained from $A'_6(\mathbb{I}^{(123)})$ and $A'_6(\mathbb{I}^{(134)})$. Those results are used in section 7.3.

First, we begin with the cut-structure given in eq. (B.1) for $A'_6(\mathbb{I}^{(123)})$. We replace $\epsilon^M \to \epsilon^L$, and use eq. (7.28). The longitudinal contributions become

$$\sum_L \frac{A'_3(1,2,P^L_{3:6}) \times A_5^{(P_{12}3)}(P^L_{12},3,4,5,6)}{s_{3:6}} = \frac{s_{1P_{3:6}}}{2\,s_{12}} \times \Bigg\{ \frac{s_{46}}{s_{4:6}} \left[s_{P_{12}P_{45}} + \frac{s_{45} s_{P_{12}6}}{s_{6P_{1:3}}} \right] + \frac{s_{6P_{34}}}{s_{56P_{12}}} \left[\frac{s_{45}\, s_{P_{12}6}}{s_{6P_{12}}} + \frac{s_{46}\, s_{P_{12}5}}{s_{6P_{34}}} + s_{P_{12}4} \right] - s_{56}\, s_{P_{12}6} \left[\frac{1}{s_{6P_{45}}} + \frac{1}{s_{6P_{12}}} \right] \Bigg\}. \tag{B.23}$$

$$\sum_L \frac{A'_5(1,2,P^L_{34},5,6) \times A_3^{(P_{5:2}3)}(P^L_{5:2},3,4)}{s_{34}} = \frac{s_{4P_{5:2}}}{2\,s_{34}} \times \Bigg\{ \frac{s_{15}}{s_{5:1}} \left[s_{P_{34}P_{16}} + \frac{s_{P_{34}5} s_{16}}{s_{5P_{2:4}}} \right] + \frac{s_{5P_{12}}}{s_{56P_{34}}} \left[\frac{s_{P_{34}5}\, s_{16}}{s_{5P_{34}}} + \frac{s_{P_{34}6}\, s_{15}}{s_{5P_{12}}} + s_{1P_{34}} \right] - s_{56}\, s_{5P_{34}} \left[\frac{1}{s_{5P_{16}}} + \frac{1}{s_{5P_{34}}} \right] \Bigg\}, \tag{B.24}$$

$$\sum_L \frac{A'_3(P^L_{4:1},2,3) \times A_5^{(1P_{23})}(1,P^L_{23},4,5,6)}{s_{4:1}} = \frac{s_{3P_{4:1}}}{2\,s_{23}} \times \Bigg\{ \frac{s_{46}}{s_{4:6}} \left[s_{P_{23}P_{45}} + \frac{s_{45} s_{P_{23}6}}{s_{6P_{1:3}}} \right] - \frac{s_{56}\, s_{P_{23}6}}{s_{6P_{45}}} + \frac{s_{6P_{2:4}}\, s_{P_{23}4}}{s_{5:1}} \Bigg\}, \tag{B.25}$$

$$\sum_L \frac{A'_4(1,2,P^L_{3:5},6) \times A_4^{(P_{6:2}3)}(P^L_{6:2},3,4,5)}{s_{3:5}} = \frac{s_{16}\, s_{6P_{3:5}}}{2\,s_{3:5}} \times \left(\frac{1}{s_{16}} + \frac{1}{s_{6P_{3:5}}} \right) \times \frac{s_{35}}{s_{3:5}} \times \left(s_{P_{6:2}P_{34}} + \frac{s_{34}\, s_{5P_{6:2}}}{s_{5P_{6:2}}} \right). \tag{B.26}$$

To end, we carry out the longitudinal contributions for all cut-contributions of $A'_6(\mathbb{I}^{(134)})$,

$$\sum_L \frac{A_3^{(1P_{3:6})}(1,2,P^L_{3:6}) \times A'_5(P^L_{12},3,4,5,6)}{s_{3:6}} = -\frac{s_{2P_{3:6}}}{2\,s_{12}} \times \Bigg\{ \frac{s_{46}}{s_{4:6}} \left[s_{P_{12}P_{45}} + \frac{s_{45} s_{P_{12}6}}{s_{6P_{1:3}}} \right] + \frac{s_{6P_{34}}}{s_{56P_{12}}} \left[\frac{s_{45}\, s_{P_{12}6}}{s_{6P_{12}}} + \frac{s_{46}\, s_{P_{12}5}}{s_{6P_{34}}} + s_{P_{12}4} \right] - s_{56}\, s_{P_{12}6} \left[\frac{1}{s_{6P_{45}}} + \frac{1}{s_{6P_{12}}} \right] \Bigg\}. \tag{B.27}$$

$$\sum_L \frac{A_5^{(1P_{34})}(1,2,P^L_{34},5,6) \times A'_3(P^L_{5:2},3,4)}{s_{34}} = -\frac{s_{4P_{5:2}}}{2\,s_{34}} \times \Bigg\{ \frac{s_{26}\, s_{56}\, s_{P_{34}5}}{s_{5P_{34}}\, s_{6P_{3:5}}} + \frac{s_{2P_{34}}\, s_{6P_{2:4}}}{s_{5:1}} + \frac{1}{s_{P_{34}56}} \left[s_{25}\, s_{P_{34}6} + \frac{s_{26}\, s_{6P_{34}}\, s_{5P_{34}}}{s_{6P_{12}}} + s_{2P_{34}}\, s_{6P_{34}} \right] \Bigg\}, \tag{B.28}$$

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$$\sum_L \frac{A_4^{(1P_{3:5})}(1,2,P^L_{3:5},6)\times A'_4(P^L_{6:2},3,4,5)}{s_{3:5}} = \frac{s_{5P_{6:2}}\, s_{45}}{s_{3:5}}\times\left(\frac{1}{s_{45}}+\frac{1}{s_{5P_{6:2}}}\right)\times \frac{s_{26}}{s_{3:5}}\times\left(s_{1P_{3:5}}+\frac{s_{12}\, s_{P_{3:5}6}}{s_{6P_{3:5}}}+s_{2P_{3:5}}\right). \tag{B.29}$$

$$\frac{A_3(3,P_{4:1},2)\times A'_5(1,P_{23},4,5,6)}{s_{4:1}} = 0, \tag{B.30}$$

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